Group Theoretical Quantization of a Phase Space $S^1 \times \mathbb{R}^+$ and the Mass Spectrum of Schwarzschild Black Holes in D Space-Time Dimensions

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Abstract

The symplectic reduction of pure spherically symmetric (Schwarzschild) classical gravity in D space-time dimensions yields a 2-dimensional phase space of observables consisting of the mass M (> 0) and a canonically conjugate (Killing) time variable T. Imposing (mass-dependent) periodic boundary conditions in time on the associated quantum mechanical plane waves which represent the Schwarzschild system in the period just before or during the formation of a black hole, yields an energy spectrum of the hole which realizes the old Bekenstein postulate that the quanta of the horizon A_{D-2} are multiples of a basic area quantum.

In the present paper it is shown that the phase space of such Schwarzschild black holes in D space-time dimensions is symplectomorphic to a symplectic manifold $\mathcal{S}=\{(\varphi \in \mathbb{R} \bmod 2\pi,\ p \ \propto A_{D-2} \in \mathbb{R}^+)\}$ with the symplectic form $d\varphi \wedge dp$. As the action of the group $SO^{\uparrow}(1,2)$ on that manifold is transitive, effective and Hamiltonian, it can be used for a group theoretical quantization of the system. The area operator $\hat{p}$ for the horizon corresponds to the generator of the compact subgroup $SO(2)$ and becomes quantized accordingly:

The positive discrete series of the irreducible unitary representations of the group $SO^{\uparrow}(1,2)$ yields an (horizon) area spectrum $\propto (k+n)$, where $k = 1, 2, \dots$ characterizes the representation and $n = 0, 1, 2, \dots$ the number of area quanta. If one employs the unitary representations of the universal covering group of $SO^{\uparrow}(1,2)$, the number k can take any fixed positive real value (θ-parameter!).

The unitary representations of the positive discrete series provide concrete Hilbert spaces for quantum Schwarzschild black holes.

PACS numbers: 03.65.Ca; 03.65.Fd; 04.60.Ds; 04.60.Kz

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1 Introduction

Understanding the quantum properties of black holes and the associated quantum statistics (thermodynamics) is one of the outstanding topics of present-day approaches to quantum gravity: In string theory extremal black holes with their relations between masses and charges and the associated degeneracies play a prominent role (for a review see Ref. [1]). In loop quantum gravity the action of the corresponding area operators on appropriate spin-network states is expected to yield information about the entropy of black holes (see the reviews [2]).

Already in 1974 Bekenstein proposed [3] a Bohr-Sommerfeld type of quantization for black holes which amounts to the simple quantum relation

$$A(n) \propto n\, l_{P,4}^2 \ , \quad n \in \mathbb{N} \equiv \{1, 2, \ldots\} \ , \tag{1}$$

($l_{P,4}$: Planck's length in $D = 4$ space-time dimensions) for the 2-dimensional area $A = 4\pi\, R_S^2$ of the horizon of a Schwarzschild black hole (SBH) in 4 space-time dimensions. Because of $R_S = 2\,G\,M/c^2$ this is equivalent to the assertion that the energy levels E_n of such holes are proportional to $\sqrt{n}$:

$$E_n \propto \sqrt{n}\, E_{P,4} \ , \quad n \in \mathbb{N} \ . \tag{2}$$

($E_{P,4}$: Planck's energy).

Together with the assumption that the n-th level (1) has a degeneracy $d_n = g^n, g > 1$, one then gets the Bekenstein-Hawking entropy of the black hole as proportional to the area of the horizon. As to the further history of the spectrum (1) and its degeneracies see the recent review by Bekenstein [4] and the Refs. [5, 6].

However, having an appropriate spectrum including its degeneracies is not sufficient for a complete quantum mechanical description of the system. For that purpose one has to know the Hilbert space and the action of the basic operators associated with the system.

It is the main aim of the present paper to provide that Hilbert space (or a number of unitarily equivalent ones) and the basic self-adjoint operators in terms of the positive discrete series of the irreducible unitary representations of the proper orthochronous Lorentz group $SO^{\uparrow}(1, 2)$ or its respective covering groups.

The method to achieve this makes use of a proposal of one of us [7] how to relate the spectrum (2) to plane wave solutions of the simple Schroedinger equation resulting from the symplectic reduction of spherically symmetric pure gravity [8, 9, 10, 11].

In order to have this paper to some extent self-contained we first summarize the essential steps and assumptions leading to the required quantum theory. As to more details and possible questions concerning these steps we refer to Ref. [7] and the companion paper [6].

The symplectic reduction of spherically symmetric pure Einstein gravity yields one pair of canonical variables ("observables" in the sense of Dirac), namely the (ADM) mass M of the system — here considered as the canonical momentum — and a canonically conjugate time variable T, with the associated symplectic form

$$\omega = dT \wedge dM \quad . \tag{3}$$

The Schroedinger equation of the corresponding quantum mechanics is extremely simple:

$$i\hbar\partial_\tau\psi(\tau) = Mc^2\psi(\tau) \quad , \tag{4}$$

where τ is the proper time of an observer at (Minkowski flat) spatial infinity.

Eq. (4) has the plane wave solutions

$$e^{-(i/\hbar)Mc^2\,\tau} \quad . \tag{5}$$

Up to now no restrictions have been imposed on the spectrum of the masses M which a priori may be any real number $M \in \mathbb{R}$. However, for physical reasons — no naked singularities etc. — one wants M to be a positive real number, $M \in \mathbb{R}^+ \equiv \{r \in \mathbb{R}, r > 0\}$.

The discrete spectrum (2) may be obtained as follows [7]: Suppose the plane wave (5) represents the system only during a finite time period Δ. Implementing this finite duration by periodic boundary conditions in τ leads to the relation

$$c^2M\,\Delta = 2\pi\hbar n,\ n \in \mathbb{N} \quad . \tag{6}$$

Here the assumption $M > 0$ is made.

It appears necessary to stress the following point in connection with the boundary condition (6): The postulate that the wave function (5) has the period Δ does not mean that the asymptotic time τ is periodic. It just means that the system is in a (quasi-) stationary state (5) during a finite time interval Δ. This is a situation completely analogous to a system of free particles in a finite spatial interval of length L where the associated state is a plane wave with periodic spatial boundary conditions (yielding discrete momenta). Such a property of the wave function does not mean that space itself is confined to an interval or periodic.

The question now is how to choose the time interval Δ. As the only intrinsic length (time) scale of the system is the Schwarzschild radius $R_S(M)$, the interval Δ has to be related to R_S/c. There are two important time scales associated with a Schwarzschild black hole, namely the "formation" time or the inverse Hawking temperature which are proportional to R_S/c and the "evaporation" time due to Hawking radiation which in 4 space-time dimensions is proportional to R_S^3 (Stefan-Boltzmann's law [12]). It appears to be more plausible [6] to associate the plane wave (5) with the quasi-stationary pre-collapse phase of the system than with the evaporation one. The assumption $\Delta = \gamma\, R_S/c\,,\ \gamma = O(1)\,$, leads to the quantization condition

$$\gamma c M_n\, R_S(M_n) = 2\pi\hbar\, n,\ \ n \in \mathbb{N}\ \ . \tag{7}$$

With $R_S = 2\, M\, G/c^2$ one immediately get the relations (1) and (2).

The relations (5)-(7) may be generalized [13, 6] to arbitrary space-time dimensions $D \geq 4$ (see also appendix B of the present paper), where the relationship between Schwarzschild mass and Schwarzschild radius is given by [14]

$$R_S^{D-3} = \frac{16\pi G_D M}{c^2(D-2)\omega_{D-2}}\ \ . \tag{8}$$

(G_D denotes the gravitational constant in D space-time dimensions and $\omega_{D-2} = 2\pi^{(D-1)/2}/\Gamma((D-1)/2)$ the volume of S^{D-2}.)

As the area A_{D-2} of the $(D-2)$-dimensional horizon is $R_S^{D-2}\omega_{D-2}$, the relations (7) and (8) imply a horizon area spectrum

$$\begin{aligned} A_{D-2}(n) &= n\,\tilde{a}_{D-2}\ ,\ n \in \mathbb{N}\ , \\ &\tilde{a}_{D-2} = \frac{32\pi^2 G_D\hbar}{\gamma(D-2)c^3} \equiv \frac{32\,\pi^2}{\gamma(D-2)}\ l_{P,D}^{D-2}\ \ , \end{aligned} \tag{9}$$

which, according to Eq. (8), leads to the energy spectrum

$$\begin{aligned} E_n &= \alpha_D\, n^{(D-3)/(D-2)} E_{P,D}\ , \\ &\alpha_D = \left(\frac{(2\pi)^{D-4}\,(D-2)\,\omega_{D-2}}{8\gamma^{D-3}}\right)^{1/(D-2)}\ , \\ &E_{P,D} = (c^{D+1}\,\hbar^{D-3}/G_D)^{1/(D-2)}\ \ . \end{aligned} \tag{10}$$

The energy E_n may be interpreted [15, 6] as the surface energy of a "bubble" of n area quanta $\tilde{a}_{D-2}$.

The above arguments, which lead to the spectra (9) and (10), respectively, are unsatisfactory for the following reasons: The period $\Delta(M)$ of

the time variable τ (or T) is a function of M. This means that the phase space of interest here is the subspace ("wedge") of the $(M,\tau)-$plane which is bounded by the positive M-axis — without the origin — and the curve $\tau = \Delta(M) = const.\ M^{1/(D-3)} > 0$, where the points M and $\Delta(M)$ have to be identified. This is an unusual phase space. In addition it is not obvious what is the Hilbert space associated with the quantized system.

It is the purpose of the present paper to improve the situation by employing a group theoretical quantization [16, 17] based on the group $SO^{\uparrow}(1,2)$ — the orthochronous proper Lorentz group in $1+2$ dimensions — and its irreducible unitary representations:

The transformation

$$\varphi = \frac{2\pi}{\Delta}\,\tau = \frac{2\pi c}{\gamma\, R_S(M)}\,\tau \ , \tag{11}$$

$$p = \beta\, A_{D-2}(M)\ ,\ \beta = \frac{\gamma c\,(D-3)}{32\pi^2 G_D}\ , \tag{12}$$

is canonical (symplectic):

$$\omega = d\varphi \wedge dp = d\tau \wedge dM \ . \tag{13}$$

As $\varphi \in \mathbb{R} \bmod 2\pi$ now, one sees that the phase space in question is diffeomorphic to $S^1 \times \mathbb{R}^+ \simeq \mathbb{R}^2 - \{0\}$. This phase space may be interpreted as "half" of the cotangent bundle $T^*S^1 = \{(\varphi,p)\}$ with the restriction $p > 0$.

The task is then to quantize this classical system appropriately. This will be done in the following way:

In section 2 we summarize, following Isham's review [16], the essential features of the group theoretical approach for quantizing a classical symplectic (phase) manifold. In section 3 we discuss the symplectic, transitive and effective action of the 3-dimensional noncompact group $SO^{\uparrow}(1,2)$ on

$$\mathcal{S} = \{(\varphi,p); \varphi \in \mathbb{R} \bmod 2\pi,\ p > 0\} \tag{14}$$

by employing the 2-fold covering group $SU(1,1)$. We show which vector fields are induced on $\mathcal{S}$ by the generators of three independent one-dimensional subgroups of $SO^{\uparrow}(1,2)$, how the isomorphism between these vector fields and the corresponding Hamiltonian ones and their Poisson algebra looks like and which observables (functions) on $\mathcal{S}$ correspond to the three selected Lie algebra elements of $SO^{\uparrow}(1,2)$. It turns out that the above variable p corresponds to the generator of the compact subgroup of $SO^{\uparrow}(1,2)$.

As this group is infinitely connected with first homotopy group $\pi_1 = \mathbb{Z}$, its covering groups, especially the universal one, act only almost effectively on $\mathcal{S}$, because the elements of their discrete abelian center leave every point of $\mathcal{S}$ fixed.

The phase space $\mathcal{S}$ is diffeomorphic to the complex plane with the origin deleted, and likewise to the coset space $SO^{\uparrow}(1,2)/\mathrm{N}$, where N is the nilpotent subgroup in an Iwasawa decomposition of $SO^{\uparrow}(1,2)$.

Quantization, which is discussed in section 4, consists in passing to the irreducible unitary representations of $SO^{\uparrow}(1,2)$ where the basic observables corresponding to three Lie algebra elements mentioned above become self-adjoint operators. The operator representing the generator of the compact subgroup, i.e. the observable p, has a discrete spectrum in all irreducible unitary representations. However, as we want $p \propto A_{D-2}$ to be positive, only the unitary representations of the "positive discrete" series are suitable for our purpose:

If we denote the operator corresponding to p by $\hat{p}$, then this operator has spectra $\propto k+n, n = 0,1,2,\ldots$, where the numbers $k = 1,2,\ldots$ characterize the different unitary representations with the Casimir operator $k(1-k)$. The main difference between the unitary representations of $SO^{\uparrow}(1,2)$ and those of its universal covering group is that for the latter the number k may be any positive real number.

The irreducible unitary representations provide the possible Hilbert spaces for the system. We discuss several concrete realizations which may be useful for future applications.

One especially interesting example is the space $L^2(\mathbb{R}^+, r^\alpha \exp{(-r)} dr)$, an orthonormal basis of which is given in terms of Laguerre's polynomials L_n^α, where $\alpha = 2k-1$ in our case. As this space is also the space of the radial wave functions for the 3-dimensional hydrogen atom, we can identify the following correspondence: for positive integers k we have $k = l+1$, where l is the angular momentum quantum number, and $n = n_r$, where $n_r \in \mathbb{N}_0 \equiv \{0,1,\ldots\}$ is the radial quantum number of the hydrogen atom! Thus the SBH wave functions with $k = 1$ correspond to the different s-wave wave functions of the hydrogen atom. Because of the obvious similarities between Coulomb's electrical and Newton's gravitational potentials this relationship may not be purely accidental.

Section 6 discusses some (preliminary) conclusions. Appendix A contains the main properties of the group $SO^{\uparrow}(1,2)$, its covering groups and those features of the irreducible unitary representations which are necessary for our purposes. Appendix B contains the symplectic reduction of spherically symmetric pure Einstein gravity in D space-time dimensions.

2 Principles of group theoretical quantization

A pedestrian way to quantize a classical system is to replace the classical Poisson brackets of observables (functions on phase space) by commutators of corresponding operators on a (Hilbert) space of states. This prescription has severe limits, however: It does not work properly for functions which are polynomials in the basic variables (q, p) of degree higher than two [17, 18] and, another possibility, the operators may not be self-adjoint (see, e.g. Ref. [19, 16]). Already Weyl pointed out very early that, in order to guarantee self-adjointness of the (two) unbounded operators Q and P, it is advisable to pass from the Heisenberg commutation relations $[P, Q] = \hbar/i$ etc. to the bounded operators

$$U(a) = e^{-iaP} \quad , \quad V(b) = e^{-ibQ} \quad , \tag{15}$$

$$U(a)\,V(b) = e^{i\hbar ab}\,V(b)\,U(a)\ , \tag{16}$$

$$U(a_1)\,U(a_2) = U(a_2)\,U(a_1)\ , \quad V(b_1)\,V(b_2) = V(b_2)\,V(b_1) \tag{17}$$

and look for continuous irreducible unitary representations of this (Heisenberg-Weyl) group which provide self-adjoint generators P and Q. Heisenberg's commutation relations may be interpreted as representing the Lie algebra of a 3-parameter group with the group law

$$(a_1, b_1, t_1) \cdot (a_2, b_2, t_2) = (a_1 + a_2, b_1 + b_2, t_1 + t_2 + \frac{1}{2}(b_1 a_2 - a_1 b_2)) \quad , \tag{18}$$

which represents a central extension of the abelian (symplectic, transitive and effective) translation group of $\mathbb{R}^2$ interpreted as the phase space (cotangent bundle) $T^*\mathbb{R}$. According to the von Neumann–Stone uniqueness theorem — see, e.g., [18, 19, 16] — all continuous irreducible unitary representations of the Heisenberg–Weyl group are unitarily equivalent to the Schroedinger representation, where the spectra of P and Q are the complete real lines $\mathbb{R}$.

Group theoretical quantization tries to generalize these properties to phase spaces (symplectic manifolds) with nontrivial topological structures. The main steps are (for more details we refer to the literature [16, 17]; we closely follow Isham's presentation [16]):

1. Given a (here finite-dimensional) symplectic space (manifold) $\mathcal{S} = \{s\}$ with a nondegenerate symplectic form ω, find a finite-dimensional Lie transformation group $\mathrm{G} = \{g\}$ of $\mathcal{S}$ which leaves the symplectic form ω invariant and which acts transitively and effectively (i.e. if

$g \cdot s = s\ \forall s$, then $g = e$ (unit element)). The latter condition may be relaxed to almost effective actions (i.e. if $g \cdot s = s\ \forall s$, then g is an element of a discrete center subgroup). The one-parameter subgroups $g(t) = \exp(-A\,t)$ of G generate vector fields $\tilde{A}$ on $\mathcal{S}$. As the transformations $g(t) \cdot s$ leave ω invariant the Lie-derivatives $L_{\tilde{A}}$ have the property $L_{\tilde{A}}\omega = 0$, which — together with $d\omega = 0$ — implies $d\,(i_{\tilde{A}}\omega) = 0$, where i_X denotes interior multiplication of an exterior form by a vector field X. The last relation means that $i_{\tilde{A}}\omega$ is a closed 1-form on $\mathcal{S}$. The corresponding vector fields $\tilde{A}$ are called "locally Hamiltonian". According to Poincaré's lemma one has locally $i_{\tilde{A}}\omega = df$, where $f(s)$ is some function on $\mathcal{S}$. If the first cohomology group $H^1(\mathcal{S};\mathbb{R})$ vanishes then $i_{\tilde{A}}\omega$ is exact and we have a (globally defined) Hamiltonian vector field which — in local canonical coordinates — has the form

$$\tilde{A} = X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} \quad . \tag{19}$$

If the vector field X can be written as the commutator of two other vector fields, $X = [X_1, X_2]$, then, because of $i_{[X_1,X_2]}\omega = d(i_{X_1} i_{X_2}\omega)$, X is Hamiltonian. This is the case for semisimple transformation groups G.

2. The relation (19) provides a map from smooth functions $f(s)$ on $\mathcal{S}$ onto Hamiltonian vector fields on $\mathcal{S}$, the kernel of which are the constant real numbers.

   As the commutator $[X_1, X_2]$ of two vector fields is again a vector field, the question is, which Hamiltonian vector field corresponds to the commutator. The answer is given by the Poisson bracket structure for functions on $\mathcal{S}$: The Poisson bracket of two functions $f_i(s), i = 1,2$, is given by

$$\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2}) = -X_{f_1}(f_2) = \frac{\partial f_1}{\partial q^i}\frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i}\frac{\partial f_2}{\partial q^i} \tag{20}$$

   and we have

$$[X_{f_1}, X_{f_2}] = -X_{\{f_1,f_2\}} \quad , \tag{21}$$

   which means that there is an homomorphism

$$f \to -X_f \tag{22}$$

   from the Lie algebra of "observables" $f(s)$ onto the Lie algebra of Hamiltonian vector fields on $\mathcal{S}$ with the (constant) real numbers as kernel.

3. We now come to a crucial point of the quantization procedure: We have — due to the (almost) effective action of the transformation group G on $\mathcal{S}$ — an isomorphism of the Lie algebra $\mathcal{L}(\mathrm{G})$ into the Hamiltonian vector fields on $\mathcal{S}$ and a homomorphism of the Lie algebra of observables f onto the Hamiltonian vector fields. What is needed, however, is the following: One wants an isomorphism between the Lie algebra $\mathcal{L}(\mathrm{G}) = \{A\}$ and the Poisson algebra of a preferred set of observables $P^A(s)$ such that

$$\tilde{A} = -X_{P^A} \;, \quad \{P^{A_1}, P^{A_2}\} = P^{[A_1,A_2]} \quad . \tag{23}$$

Such an isomorphism — a so-called "momentum map" — is not always possible, due to the fact that the constant functions $\in \mathcal{S}$ have vanishing Hamiltonian vector fields (Poisson brackets). If, however, the second cohomology group $H^2(\mathcal{L}(\mathrm{G}); \mathbb{R})$ vanishes, the momentum map does exist. This is the case for semisimple Lie groups, like our $SO^{\uparrow}(1,2)$. If the second cohomology group is not trivial, one may be forced to look for appropriate central extensions of the original group, like in the case of the Heisenberg group, which represents a central extension of the abelian translation group.

4. Having established an isomorphism between the Lie algebra $\mathcal{L}(\mathrm{G})$ and a corresponding Poisson algebra of a system $\{P^A\}$ of preferred observables on $\mathcal{S}$, one then can quantize the classical system by using the irreducible unitary representations of the transformation group G where the self-adjoint generators $K(A)$ of the 1-parameter subgroups $U(g(t) = \exp(At)) = \exp(-iK(A))$ represent the corresponding original classical observables P^A.

5. As there may be different groups with symplectic, transitive and effective action on $\mathcal{S}$, one has to make a choice which one to use. Here physical considerations come into play: One wants a group such that the corresponding observables $P^A(s)$ constitute basic functions on $\mathcal{S}$ so that all physically interesting observables can be expressed by them. For additional discussions of these problems see Ref. [20]

3 The action of the group $SO^{\uparrow}(1,2)$ on $S^1 \times \mathbb{R}^+$

The local versions (3) and (13) of the symplectic form ω may belong to different global geometries and, accordingly, to different ensuing quantum

theories [16]. If we have the usual phase space $T^*\mathbb{R} \simeq \mathbb{R}^2$ then the "quantizing" group is the abelian translation group of $\mathbb{R}^2$ enlarged by a central extension as described at the beginning of the preceding section.

If the phase space has the global form $T^*\mathbb{R}^+ = \{(q,p); q > 0, p \in \mathbb{R}\}$ the quantizing group G is the affine group $\mathrm{G} = \{g(a,t)\,;\ a,t \in \mathbb{R}\,;\ g(a_2,t_2)\cdot g(a_1,t_1) = g(a_2 + e^{-t_2}a_1, e^{t_1+t_2})\}$ with action $g(a,t)\cdot(q,p) = (e^t\, q, e^{-t}\, p - a)$. The self-adjoint generator $K(\mathrm{S})$ of the scale transformations $g(0,t)$ here corresponds to the classical observable $q\,p$.

Closer "home" to our system (14) is the phase space $T^*S^1 = \{(\varphi,p)\,;\ \varphi \in \mathbb{R} \bmod 2\pi,\ \ p \in \mathbb{R}\}$ for which Isham discusses in detail the 3-parametric euclidean group $\mathrm{G} \equiv E_2 = \{g(a_1,a_2,\theta); \theta \in \mathbb{R} \bmod 2\pi, a_1, a_2 \in \mathbb{R}\}$ of $\mathbb{R}^2$ with the action $g(a_1,a_2,\theta)\cdot(\varphi,p) = ((\varphi+\theta) \bmod 2\pi,\ p + a_1\sin(\varphi+\theta) - a_2\cos(\varphi+\theta))$ as quantizing group. Details can be found in Ref. [16].

The phase space of the last example is still a cotangent bundle which is no longer so in our case (14), where we have $p > 0$. In that case the orthochronous proper Lorentz group $SO^{\uparrow}(1,2)$ — which leaves the quadratic form $(x^0)^2 - (x^1)^2 - (x^2)^2\ , x^0 > 0$, invariant — appears to be the appropriate quantizing group (see also Ref. [20]): The cone $(x^0)^2 - (x^1)^2 - (x^2)^2 = 0\ ,\ x^0 > 0$, is diffeomorphic to $\mathbb{R}^2 - \{0\}$: put $x^0 = p > 0,\ x^1 = p\cos\varphi,\ x^2 = p\sin\varphi$!

In the following it is advantageous to employ the twofold covering group $SU(1,1)$ of $SO^{\uparrow}(1,2)$ (see appendix A) the elements g_0 of which are given by

$$g_0 = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}\ ,\quad |\alpha|^2 - |\beta|^2 = 1\ , \tag{24}$$

where $\bar{\alpha}$ means the complex conjugate of the complex number α. If we define the matrix

$$X_0 = \begin{pmatrix} x^0 & x^1 - i\,x^2 \\ x^1 + i\,x^2 & x^0 \end{pmatrix}\ ,\quad \det X_0 = (x^0)^2 - (x^1)^2 - (x^2)^2\ \ , \tag{25}$$

then the transformations $x^\mu \to \hat{x}^\mu,\ \mu = 0,1,2$, under $SO^{\uparrow}(1,2)$ are implemented by

$$X_0 \to \hat{X}_0 = g_0 X_0 g_0^+\ ,\quad \det\hat{X}_0 = \mathrm{X}_0\ \ , \tag{26}$$

where g_0^+ denotes the hermitian conjugate of the matrix g_0.

Applying a general transformation g_0 to the matrix

$$\begin{pmatrix} p & p\,e^{-i\varphi} \\ p\,e^{i\varphi} & p \end{pmatrix} \tag{27}$$

yields the mapping: $(p,\varphi) \to (\hat{p},\hat{\varphi})$,

$$\hat{p} \;=\; |\alpha + e^{i\varphi}\,\beta|^2\, p \quad , \tag{28}$$

$$e^{i\hat{\varphi}} \;=\; \frac{\bar{\alpha}e^{i\varphi} + \bar{\beta}}{\alpha + e^{i\varphi}\,\beta} \quad . \tag{29}$$

As

$$\frac{\partial\hat{\varphi}}{\partial\varphi} = |\alpha + e^{i\varphi}\beta|^{-2} \tag{30}$$

we have

$$d\hat{\varphi} \wedge d\hat{p} = d\varphi \wedge dp \quad , \tag{31}$$

that is, the transformations (28) and (29) are symplectic.

One sees immediately that g_0 and $-g_0$ lead to the same transformations of p and φ. Thus, the group $SU(1,1)$ acts on $\mathcal{S}$ only almost effectively with kernel $\mathbb{Z}_2$ representing the center of the twofold covering group of $SO^{\uparrow}(1,2)$. It is well-known that the latter group acts effectively and transitively on the forward light cone and thus on $\mathcal{S}$ (see also below).

We next discuss the actions of the 1-parametric subgroups $\mathrm{K}_0, \mathrm{A}_0$ and N_0 forming the Iwasawa decomposition $SU(1,1) = \mathrm{K}_0 \cdot \mathrm{A}_0 \cdot \mathrm{N}_0$, with the general element

$$\begin{aligned} k_0 \cdot a_0 \cdot n_0 \;&=\; \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \cdot \begin{pmatrix} \cosh(t/2) & -i\,\sinh(t/2) \\ i\,\sinh(t/2) & \cosh(t/2) \end{pmatrix} \\ &\cdot \begin{pmatrix} 1 - i\xi/2 & \xi/2 \\ \xi/2 & 1 + i\,\xi/2 \end{pmatrix} \ , \end{aligned} \tag{32}$$

where $\theta \in (-2\pi, +2\pi]$; $t, \xi \in \mathbb{R}$.

The actions of the subgroups $\mathrm{K}_0, \mathrm{A}_0, \mathrm{N}_0$, respectively, are the following ones:

$$\mathrm{K}_0 : \qquad \hat{p} = p \ , \qquad e^{i\hat{\varphi}} = e^{i(\varphi+\theta)} \quad . \tag{33}$$

$$\mathrm{A}_0 : \qquad \hat{p} = \rho(t,\varphi)\, p \ , \quad \rho(t,\varphi) = \cosh t + \sinh t\, \sin\varphi \ , \tag{34}$$

$$\cos\hat{\varphi} = \cos\varphi/\rho(t,\varphi) \ , \quad \sin\hat{\varphi} = (\cosh t \sin\varphi + \sinh t)/\rho(t,\varphi) \quad .$$

$$\mathrm{N}_0 : \qquad \hat{p} = \rho(\xi,\varphi)\, p \ , \quad \rho(\xi,\varphi) = 1 + \xi\cos\varphi + \xi^2(1-\sin\varphi)/2 \ , \tag{35}$$

$$\cos\hat{\varphi} = [\cos\varphi + \xi(1-\sin\varphi)]/\rho(\xi,\varphi) \ ,$$

$$\sin\hat{\varphi} = [\sin\varphi + \xi\cos\varphi + \xi^2(1-\sin\varphi)/2]/\rho(\xi,\varphi) \quad .$$

The groups (33) and (34) act transitively on $\mathcal{S}$: Any point $s_1 = (\varphi_1, p_1)$ may be transformed into any other point $s_2 = (\varphi_2, p_2)$ in the following way: first transform (φ_1, p_1) into $(0, p_1)$ by $k_0(\theta = -\varphi_1)$, then map this point into $(\hat{\varphi} = \arctan(\sinh \hat{t}\,), p_2)$ by $a_0(\hat{t};\, \cosh \hat{t} = p_2/p_1)$ and finally transform $(\hat{\varphi}, p_2)$ by $k_0(\theta = \varphi_2 - \hat{\varphi})$ into $s_2 = (\varphi_2, p_2)$. As K_0 and A_0 combined act already transitively on $\mathcal{S}$ one might wonder whether they alone are not sufficient for our purpose. However, they do not form a 2-dimensional subgroup of $SU(1,1)$, only A_0 and N_0 do. The above transitivity properties reflect the fact that any element g_0 of $SU(1,1)$ may be written as $k_0(\theta_2) \cdot a_0(t) \cdot k_0(\theta_1)$ (see appendix A).

The transformation formulae (35) show that the group N_0 leaves the half-line $\varphi = \pi/2$, $p > 0$ invariant, that is, N_0 is the stability group of these points. This means that the symplectic space (14) is diffeomorphic to the coset space $SU(1,1)/(\mathbb{Z}_2 \times \mathrm{N}_0) \simeq SO^{\uparrow}(1,2)/\mathrm{N}_0$. Notice that N_0, and A_0 as well, does not contain the second center element $-e$ of $SU(1,1)$. The center $\mathbb{Z}_2$ is a subgroup of K_0.

If we pass to the universal covering group $\widetilde{SU(1,1)}$ of $SU(1,1)$ (or of $SO^{\uparrow}(1,2)$), see Eq. (24),

$$\widetilde{SU(1,1)} = \{\tilde{g} = (\omega, \gamma);\ \omega \equiv \arg(\alpha) \in \mathbb{R},\ \gamma = \beta/\alpha,\ |\gamma| < 1)\}\ , \tag{36}$$

(as to the group multiplication laws see appendix A), the transformations (28) and (29) take the form

$$\hat{p} = \rho(\tilde{g}, \varphi)\, p\ ,\quad \rho(\tilde{g}, \varphi) = |1 + e^{i\varphi}\,\gamma|^2\,(1 - |\gamma|^2)^{-1}\ , \tag{37}$$

$$e^{i\hat{\varphi}} = e^{-2i\omega}\,\frac{e^{i\varphi} + \bar{\gamma}}{1 + e^{i\varphi}\gamma}\ . \tag{38}$$

As $\partial\hat{\varphi}/\partial\varphi = 1/\rho(\tilde{g}, \varphi)$, the equality (31) holds again.

With the elements of the group $SU(1,1)$ given by the restriction $-\pi < \omega \leq +\pi,\ \alpha = \exp(i\omega)(1 - |\gamma|^2)^{-1/2},\ \beta = \gamma\,\alpha$, the homomorphisms

$$h\ :\ \widetilde{SU(1,1)} \to SU(1,1)\ , \tag{39}$$

$$h_0\ :\ SU(1,1) \to SO^{\uparrow}(1,2)\ , \tag{40}$$

have the kernels $\ker(h) = 2\pi\,\mathbb{Z}$, $\ker(h_0) = \mathbb{Z}_2$, respectively, and the composite homomorphism $h_0 \circ h$ has the kernel $\pi\mathbb{Z}$.

As the space $\mathcal{S}$, Eq. (14), is diffeomorphic to $\mathbb{R}^2 - \{0\} = \mathbb{C} - \{0\}$, its universal covering space is given by $\varphi \in \mathbb{R},\ p \in \mathbb{R}^+$ which is the infinitely sheeted Riemann surface of the logarithm. The transformations (37) and

(38) may be interpreted as acting transitively and effectively on that universal covering space.

We would like to mention that one can define genuine *effective* actions of any covering group of $SO^{\uparrow}(1,2)$ on $\mathcal{S}$. However, these actions violate the "strong generating principle" of Isham [16] and are not adequate for a group theoretical quantization [20].

The action of $SO^{\uparrow}(1,2)$ on $\mathcal{S}$ may also be obtained as a lift of the respective subgroup of Diff(S^1) to $T^*S^1 \supset \mathcal{S}$. This will be discussed further in Ref. [20], where it is also shown that, under certain conditions, the group $SO^{\uparrow}(1,2)$ is unique as to the required action on $\mathcal{S}$.

4 Hamiltonian vector fields induced on $\mathcal{S}$ by $SU(1,1)$ transformations and the corresponding classical observables

For infinitesimal values of the parameters θ, t, ξ the transformations (33)-(35) take the form

$$\mathrm{K}: \quad \delta\varphi = \theta,\ |\theta| \ll 1, \quad \delta p = 0\ , \tag{41}$$

$$\mathrm{A}: \quad \delta\varphi = (\cos\varphi)\, t, \quad \delta p = p\,(\sin\varphi)\, t, \quad |t| \ll 1\ , \tag{42}$$

$$\mathrm{N}: \quad \delta\varphi = (1-\sin\varphi)\,\xi, \quad \delta p = p\,(\cos\varphi)\,\xi, \quad |\xi| \ll 1\ . \tag{43}$$

They induce on $\mathcal{S}$ the vector fields

$$\tilde{A}_K = -\partial_\varphi\ , \tag{44}$$

$$\tilde{A}_A = -\cos\varphi\,\partial_\varphi - p\,\sin\varphi\,\partial_p\ , \tag{45}$$

$$\tilde{A}_N = (\sin\varphi - 1)\partial_\varphi - p\,\cos\varphi\,\partial_p\ . \tag{46}$$

It follows from the general considerations of the preceding section and it is easy to check that their Lie algebra is isomorphic to the Lie algebra of $SO^{\uparrow}(1,2)$ (see appendix A) (and all its covering groups, of course). A general element is the linear combination

$$\begin{aligned} \tilde{A}(\lambda_K,\lambda_A,\lambda_N) &= \lambda_K\tilde{A}_K + \lambda_A\tilde{A}_A + \lambda_N\tilde{A}_N \\ &= -(\lambda_A\,p\,\sin\varphi + \lambda_N\,p\,\cos\varphi)\partial_p \\ &\quad -(\lambda_K + \lambda_A\,\cos\varphi + \lambda_N\,(1-\sin\varphi))\partial_\varphi\ . \end{aligned} \tag{47}$$

One sees immediately that this vector field can be identified with the Hamiltonian vector field

$$-X_f = \frac{\partial f}{\partial\varphi}\partial_p - \frac{\partial f}{\partial p}\partial_\varphi\ , \tag{48}$$

$$f(\varphi,p) \;=\; \lambda_K\,p+\lambda_A\,p\,\cos\varphi+\lambda_N\,p\,(1-\sin\varphi)\;\;. \tag{49}$$

If we replace the Lie algebra element l_N by $l_B = l_N - l_K$ (see appendix A), then the observable f becomes

$$f(\varphi,p)=\lambda_K\,p+\lambda_A\,p\,\cos\varphi-\lambda_B\,p\,\sin\varphi\;\;, \tag{50}$$

and we see that the associated 3 basic classical observables are

$$P^K = p,\quad P^A = p\,\cos\varphi,\quad P^B = -p\,\sin\varphi\;\;. \tag{51}$$

As any smooth function $f(\varphi,p)$ periodic in φ with period 2π can, under certain conditions, be expanded in a Fourier series and as $\sin(n\varphi)$ and $\cos(n\varphi)$ can be expressed as polynomials of n-th order in $\sin\varphi$ and $\cos\varphi$, the observables (51) are indeed basic ones on $\mathcal{S}$. Actually they are just the Cartesian coordinates of $\mathbb{R}-\{0\}$ we started with, see Eq. (27).

5 Quantum spectrum of the area operator and associated Hilbert spaces

We now come to the quantization of the classical system we have been discussing. It consists in replacing each of the three basic observables (51) by the self-adjoint generator $K(l)$ of the unitary operator $U_l(t)=\exp(-i\,K(l)\,t)$ (or $\exp(i\,K(l)\,t)$), representing any of the associated 1-parameter subgroups $\exp(l\,t), l\in\mathcal{L}SO^{\uparrow}(1,2)$, in an appropriate irreducible unitary representation of $SO^{\uparrow}(1,2)$ or its covering groups: Thus, the observable p is to be replaced by the self-adjoint generator $K_3 \equiv K(l_K)$ of the unitary operator $U(\theta) = \exp(-iK_3\,\theta)$ representing the compact subgroup $\mathrm{K}=\{\exp(l_K\,\theta)\}$ and the observables $p\cos\varphi$ and $-p\sin\varphi$ are to be replaced by the corresponding self-adjoint generators K_1 and K_2 of the unitary operators $U(t_1)$ and $U(t_2)$ representing the ("boost") subgroups A and B.

In this section we mainly use the known properties of the irreducible unitary representations of $SO^{\uparrow}(1,2)$ and $\widetilde{SO^{\uparrow}(1,2)}$. More details and references to the literature are contained in appendix A.

We first put $\hbar=1$ and restore it explicitely later.

Because K_3 is associated with a compact subgroup, its spectrum is discrete in all irreducible unitary representations of $SO^{\uparrow}(1,2)$ or its covering groups. However, not all irreducible unitary representations are suitable for our purposes, because $\hat{p}=K_3$ corresponds to a classical area, Eq. (12), which is positive. Thus, we are interested in those irreducible representations for

which the spectrum of K_3 is positive. This is the case for the so-called "positive discrete series" of irreducible unitary representations. These representations are characterized by the value $k(1-k)$ of the associated Casimir operator $Q = K_1^2 + K_2^2 - K_3^2$, where k can take the values $1, 2, \ldots$, for a "true" representation of $SO^{\uparrow}(1,2)$, but k can be any positive real number > 0 for the corresponding representations of the universal covering group $\widetilde{SO^{\uparrow}(1,2)}$. For the groups $SU(1,1) \simeq SL(2,\mathbb{R})$ the number k can take the values $1/2, 1, 3/2, 2, \ldots$. In all cases the operator $\hat{p} = K_3$ has the spectrum

$$\mathrm{spec}\,(\hat{p} \equiv K_3) = \{k + n,\, n \in \mathbb{N}_0\} \quad . \tag{52}$$

For the representations of the universal covering group k mod 1 represents the so-called "θ-parameter" which occurs in other unitary representations involving the infinitely sheeted compact group $SO(2)$ [16, 21]. For a more general setting of that parameter in connection with multiply connected symplectic manifolds see again Ref. [16].

As only $SO^{\uparrow}(1,2)$ acts effectively on the symplectic space $\mathcal{S}$ of Eq. (14), the θ-parameter comes into play merely if we allow for almost effective group actions by the universal covering group. Whether one has to do so or not, finally has to be decided by physical considerations.

We next come to the discussion of concrete Hilbert spaces on which the operators K_i, $i = 1, 2, 3$, act as self-adjoint operators and where K_3 has one of the spectra (52).

5.1 Hilbert space of holomorphic functions inside the unit disc $\mathcal{D}$

Probably the most important Hilbert space is the (Bargmann) Hilbert space $\mathcal{H}_{\mathcal{D},k}$ of holomorphic functions in the unit disc $\mathcal{D} = \{z = x + iy,\, |z| < 1\}$ with the scalar product

$$(f,g)_{\mathcal{D},k} = \frac{2k-1}{\pi} \int_{\mathcal{D}} \bar{f}(z) g(z)\, (1 - |z|^2)^{2k-2} dx dy \quad . \tag{53}$$

It can be used for any real $k > 1/2$ and also in the limiting case $k \to 1/2$. As

$$(z^{n_1}, z^{n_2})_{\mathcal{D},k} = \frac{\Gamma(2k)\,\Gamma(n_1+1)}{\Gamma(2k+n_1)}\, \delta_{n_1 n_2} \ , \tag{54}$$

and since any holomorphic function in $\mathcal{D}$ can be expanded in powers of z the functions

$$e_{k,n}(z) = \sqrt{\frac{\Gamma(2k+n)}{\Gamma(2k)\,\Gamma(n+1)}}\, z^n \quad , \quad n \in \mathbb{N}_0 \ , \tag{55}$$

form an orthonormal basis of $\mathcal{H}_{\mathcal{D},k}$. The operators K_i here have the explicit forms

$$
\begin{aligned}
K_3 &= k + z\frac{d}{dz} \quad , & (56)\\
K_+ &\equiv K_1 + i\,K_2 = 2kz + z^2\frac{d}{dz} \quad , & (57)\\
K_- &\equiv K_1 - iK_2 = \frac{d}{dz} \quad . & (58)
\end{aligned}
$$

The basis functions (55) are the eigenfunctions of K_3 with eigenvalues $k+n$, the operators K_+ and K_- being raising and lowering operators:

$$
\begin{aligned}
K_3\, e_{k,n} &= (k+n)\, e_{k,n} \quad , & (59)\\
K_+\, e_{k,n} &= [(2k+n)(n+1)]^{1/2}\, e_{k,n+1} \quad , & (60)\\
K_-\, e_{k,n} &= [(2k+n-1)n]^{1/2}\, e_{k,n-1} \quad . & (61)
\end{aligned}
$$

The formulae (1) and (9) suggest to associate them with the irreducible representation $k=1$, that is with scalar product and eigenfunctions

$$
(f,g) = \frac{1}{\pi}\int_{\mathcal{D}} dxdy\bar{f}(z)g(z) \ , \quad e_{1,n} = \sqrt{n+1}\; z^n, \ n \in \mathbb{N}_0 \ . \qquad (62)
$$

If we have on $\mathcal{D}$ the holomorphic functions

$$
f(z) = \sum_{n=0}^{\infty} a_n\, z^n \ , \quad g(z) = \sum_{n=0}^{\infty} b_n\, z^n \ , \qquad (63)
$$

then, according to Eq. (54), their scalar product $(f,g)_{\mathcal{D},k}$ is given by

$$
(f,g)_{\mathcal{D},k} = \sum_{n=0}^{\infty} \frac{\Gamma(2k)\,\Gamma(n+1)}{\Gamma(2k+n)} \bar{a}_n\, b_n \quad . \qquad (64)
$$

This series can be used as a scalar product to extend the definition of the Hilbert spaces $\mathcal{H}_{\mathcal{D},k}$ to all real $k > 0$!

5.2 The Hardy space of the unit circle

For the special case $k = 1/2$ the coefficient in front of $\bar{a}_n\, b_n$ in (64) has the value 1. This allows for a reinterpretation of the Hilbert space $\mathcal{H}_{\mathcal{D},\frac{1}{2}}$: Consider the L^2-space on the unit circle with the scalar product

$$
(\psi_1, \psi_2) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\; \bar{\psi}_1(\phi)\, \psi_2(\phi) \quad , \qquad (65)
$$

an orthonormal basis of which is given by the functions $\exp(in\phi)$, $n \in \mathbb{Z}$. That subspace of functions $h(\phi) \in L^2$ which have only "positive" Fourier coefficients, $a_n = 0$, $n < 0$, is called the "Hardy space H^2_+ of the unit circle", and the corresponding scalar product is denoted by $(h_1, h_2)_+$. It has the orthonormal basis $\exp(in\phi)$, $n \in \mathbb{N}_0$.

Hardy spaces [22, 23, 24] have a number of interesting properties and are closely related to Hilbert spaces of holomorphic functions [25, 22, 23, 24].

If we have the two Fourier series $\in H^2_+$

$$h_1(\phi) = \sum_{n=0}^{\infty} a_n \, e^{in\phi} \ , \quad h_2(\phi) = \sum_{n=0}^{\infty} b_n \, e^{in\phi} \quad , \tag{66}$$

they have the scalar product

$$(h_1, h_2)_+ = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, \bar{h}_1(\phi) h_2(\phi) = \sum_{n=0}^{\infty} \bar{a}_n \, b_n \quad . \tag{67}$$

Thus we may realize the Hilbert space $\mathcal{H}_{\mathcal{D},\frac{1}{2}}$ by using the Hardy space H^2_+!

5.3 Unitary representations on Hardy space related Hilbert spaces

What is the relation of the other spaces $\mathcal{H}_{\mathcal{D},k}$ to the Hardy space H^2_+? The answer is somewhat subtle [26, 27, 57]: Define the self-adjoint operator A_k in H^2_+ which is diagonal in the basis $\{\exp(in\phi)\}$ of H^2_+ and which acts on it as

$$A_k \, e^{in\phi} = \frac{\Gamma(2k)\,\Gamma(n+1)}{\Gamma(2k+n)} \, e^{in\phi} \quad . \tag{68}$$

Then define an H^2_+ related Hilbert space $H^2_{A_k}$ with the scalar product

$$(h_1, h_2)_k = (h_1, A_k \, h_2) = \sum_{n=0}^{\infty} \frac{\Gamma(2k)\,\Gamma(n+1)}{\Gamma(2k+n)} \bar{a}_n \, b_n \tag{69}$$

for the functions (66). The series (69) representing the scalar product of $H^2_{A_k}$ is obviously the same as the series (64) which represents the scalar product for $\mathcal{H}_{\mathcal{D},k}$. This exhibits the very close relationship between the two Hilbert spaces. The mathematical background for this is that holomorphic functions inside the unit disc $\mathcal{D}$ have holomorphic limits on $\partial\mathcal{D} = S^1$ (for

mathematical details see the Refs. [22, 23, 24]).
An orthonormal basis for $H^2_{A_k}$ is given by

$$\chi_{k,n}(\phi) = \sqrt{\frac{\Gamma(2k+n)}{\Gamma(2k)\,\Gamma(n+1)}}\, e^{i(k+n)\phi},\ n \in \mathbb{N}_0 \ , \qquad (70)$$

$$(\chi_{k,n_1}, \chi_{k,n_2})_k = \delta_{n_1 n_2} \quad ,$$

where we have included an overall phase factor $\exp(ik\phi)$.

With respect to this basis the operators K_3, K_+, K_- have the form

$$K_3 = \frac{1}{i}\,\partial_\phi \ , \quad K_+ = e^{i\phi}(ik + \partial_\phi) \ , \quad K_- = e^{-i\phi}(ik - \partial_\phi) \quad . \qquad (71)$$

Their action on the basis functions (70) is given by

$$K_3 \chi_{k,n} = (k+n)\chi_{k,n} \quad , \qquad (72)$$

$$K_+ \chi_{k,n} = i[(2k+n)(n+1)]^{1/2} \chi_{k,n+1} \quad , \qquad (73)$$

$$K_- \chi_{k,n} = \frac{1}{i}[(2k+n-1)n)]^{1/2} \chi_{k,n-1} \quad . \qquad (74)$$

It is important to realize that the operators K_3, K_+, K_- belong to a representation which is unitary only with respect to the scalar product (69), not with respect to the scalar product (67)! This may be seen explicitly as follows: Applying the operators K_+ and K_- from Eq. (71) to the series

$$f_1(\phi) = \sum_{m=0}^{\infty} a_m\, \chi_{k,m}(\phi) \ , \quad f_2(\phi) = \sum_{n=0}^{\infty} b_n\, \chi_{k,n}(\phi) \quad , \qquad (75)$$

using the relations (73) and (74) and the orthonormality (70) yields

$$(f_2, K_+ f_1)_k = \sum_{n=0}^{\infty} i[(2k+n)(n+1)]^{1/2}\, \bar{b}_{n+1}\, a_n = (K_- f_2, f_1)_k \quad , \qquad (76)$$

which says that K_- is the adjoint operator of K_+ with respect to the scalar product (69). But one sees immediately that this is not so with respect to the scalar product (67)!

Furthermore, the multiplication operator $\exp(i\phi)$ is not a unitary operator on H^2_+, because its inverse does not always exist: for instance, the constant function $f = 1$ is an element of H^2_+, but $\exp(-i\phi)\cdot 1 = \exp(-i\phi)$ is not. Such isometric operators are called "shift operators" and their properties have been investigated systematically by the mathematicians [28, 22, 23, 24].

The question is, whether there are irreducible unitary representations of the positive discrete series of $SO^{\uparrow}(1,2)$ or its covering groups on the Hardy space H^2_+ itself? Sally has shown [51], by a detour, that there are such representations on H^2_+ and that they are unitarily equivalent to the ones above. We shall briefly indicate how this works, because on the way we learn about other interesting Hilbert spaces on which some of the above irreducible representations are realized.

5.4 Unitary representations in the Hilbert space of holomorphic functions on the upper half plane

The unit disc $\mathcal{D}$ and its associated Hilbert space with the scalar product (53) is especially suited for the construction of unitary representations of $SU(1,1)$ because that group acts transitively on $\mathcal{D}$ (see appendix A). Similarly, the group $SL(2,\mathbb{R})$, which is isomorphic to $SU(1,1)$, see appendix A, acts transitively on the upper complex half plane $\mathbb{C}^{+i} = \{w = u+iv,\ v > 0\}$. The mapping

$$w = \frac{1-iz}{z-i} = \frac{2x + (1-x^2-y^2)\,i}{x^2+(y-1)^2} , \tag{77}$$

$$z = \frac{iw+1}{w+i} , \quad |z|^2 = \frac{u^2+(v-1)^2}{u^2+(v+1)^2} , \tag{78}$$

provides a holomorphic diffeomeorphism from $\mathcal{D}$ onto $\mathbb{C}^{+i}$ and back. Because of

$$\frac{dudv}{4v^2} = \frac{dxdy}{(1-|z|^2)^2} , \quad 1-|z|^2 = \frac{2^2\,v}{(w+i)(\bar{w}-i)} , \tag{79}$$

we have for $k = 1/2, 1, 3/2, 2, \ldots$ the following isomorphism:

$$(f,g)_{\mathcal{D},k} = (\tilde{f},\tilde{g})_{\mathbb{C}^{+i},k} = \frac{1}{\Gamma(2k-1)} \int_{\mathbb{C}^{+i}} \overline{\tilde{f}}\,\tilde{g}\,v^{2k-2} dudv , \tag{80}$$

where

$$E_k : \qquad \tilde{f}(w) = \sqrt{\frac{\Gamma(2k)}{\pi}}\,2^{2k-1}(w+i)^{-2k} f\left(z = \frac{1+iw}{i+w}\right) , \tag{81}$$

$$E_k^{-1} : \qquad f(z) = 2\sqrt{\frac{\pi}{\Gamma(2k)}}\,(z-i)^{-2k}\tilde{f}\left(w = \frac{1-iz}{z-i}\right) . \tag{82}$$

The (unitary) transformation E_k maps the basis (55) of $\mathcal{H}_{\mathcal{D},k}$ onto the basis

$$\tilde{e}_{k,n}(w) = \sqrt{\frac{\Gamma(2k+n)}{\pi\Gamma(n+1)}}\,2^{2k-1}\,i^n\,(w-i)^n\,(w+i)^{-2k-n} , n \in \mathbb{N}_0 , \tag{83}$$

of $\mathcal{H}_{\mathbb{C}^{+i},\,k}$.
One can, of course, discard the phase factor i^n.
On this Hilbert space the irreducible unitary representations T_k^+ of the positive discrete series of $SL(2,\mathbb{R})$ are given by

$$
\begin{aligned}
[T^+(g_1,k)\tilde{f}](w) &= (a_{12}w+a_{22})^{-2k}\tilde{f}\left(\frac{a_{11}w+a_{21}}{a_{12}w+a_{22}}\right) , && (84)\\
g_1 &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL(2,\mathbb{R}) , && (85)
\end{aligned}
$$

which is defined for $k = 1/2, 1, 3/2, 2, \ldots$ only. The subgroups

$$
\begin{aligned}
\mathrm{K}_1 : &\quad k_1 = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} , && (86)\\
\mathrm{A}_1 : &\quad a_1 = \begin{pmatrix} e^{t_1/2} & 0 \\ 0 & e^{-t_1/2} \end{pmatrix} , && (87)\\
\mathrm{B}_1 : &\quad b_1 = \begin{pmatrix} \cosh(t_2/2) & \sinh(t_2/2) \\ \sinh(t_2/2) & \cosh(t_2/2) \end{pmatrix} && (88)
\end{aligned}
$$

are associated with the following generators of their unitary representations (we choose the sign of $\tilde{K}_3$ such that its spectrum is positive):

$$
\begin{aligned}
\tilde{K}_3 &= \frac{1}{i}(k\,w + \frac{1}{2}(w^2+1)\frac{d}{dw}) , && (89)\\
\tilde{K}_\pm &= \pm k\,(w \mp i) \pm \frac{1}{2}(w \mp i)^2\frac{d}{dw} . && (90)
\end{aligned}
$$

Their action on the basis (83) is given by

$$
\begin{aligned}
\tilde{K}_3\tilde{e}_{k,n} &= (k+n)\tilde{e}_{k,n} , && (91)\\
\tilde{K}_+\tilde{e}_{k,n} &= i[(2k+n)(n+1)]^{1/2}\tilde{e}_{k,n+1} , && (92)\\
\tilde{K}_-\tilde{e}_{k,n} &= \frac{1}{i}[(2k+n-1)n)]^{1/2}\tilde{e}_{k,n-1} . && (93)
\end{aligned}
$$

For the limiting case $k \to 1/2$ the Hilbert space with the scalar product (80) now can be replaced by the "Hardy space H^2_{+i} of the upper half plane", [22, 23, 24] the elements of which are the functions $\tilde{f}(u)$ which are limits for $\Im(w) = v \to 0$ of the previous holomorphic functions $\tilde{f}(w)$ on the upper half plane and the Hilbert space of which has the scalar product

$$
(\tilde{f}_1,\tilde{f}_2)_{+i} = \int_{-\infty}^{\infty} du\overline{\tilde{f}}_1(u)\,\tilde{f}_2(u) . \qquad (94)
$$

5.5 Hilbert space of the Fourier transformed holomorphic functions on the upper half plane

We now pass to still another Hilbert space $\widehat{\mathcal{H}}_{\mathbb{C}^{+i},\,k}$ by the Fourier transform

$$\mathcal{F}: \qquad \hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(w) e^{-itw} du \ , \quad \tilde{f}(w) \in \mathcal{H}_{\mathbb{C}^{+i},\,k} \ , \quad t \in \mathbb{R} \ . \quad (95)$$

Because of the analyticity properties of $\tilde{f}$ one has [29] $\partial_v \hat{f}(t) = 0$ and $\hat{f}(t) = 0$ for $t < 0$ and the inversion is given by

$$\mathcal{F}^{-1}: \qquad \tilde{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \hat{f}(t) e^{iwt} dt \ . \quad (96)$$

The scalar product induced on $\widehat{\mathcal{H}}_{\mathbb{C}^{+i},\,k}$ is

$$(\tilde{f}, \tilde{g})_k = (\hat{f}, \hat{g})_k = \frac{1}{2^{2k-1}} \int_{0}^{\infty} \overline{\hat{f}}(t) \hat{g}(t) t^{1-2k} dt \ . \quad (97)$$

The Fourier transform (95) maps the basis (83) of $\mathcal{H}_{\mathbb{C}^{+i},\,k}$ onto the basis [30]

$$\hat{e}_{k,n}(t) = i^{n-2k} \sqrt{\frac{2\Gamma(n+1)}{\Gamma(2k+n)}} (2t)^{2k-1} e^{-t} L_n^{2k-1}(2t) \quad (98)$$

of $\widehat{\mathcal{H}}_{\mathbb{C}^{+i},\,k}$. Here L_n^{2k-1} are Laguerre's polynomials which obey the equation [31]

$$x\, L_n^{2k-1\,\prime\prime} + (2k - x) L_n^{2k-1\,\prime} + n\, L_n^{2k-1} = 0 \ . \quad (99)$$

Using the inverse Fourier transform (96) the operator $\tilde{K}_3$ can be seen to take now the form

$$\hat{K}_3 = -\frac{1}{2} \frac{d^2}{dt^2} t + k \frac{d}{dt} + \frac{t}{2} \ , \quad (100)$$

of which the basis functions (98) are eigenfunctions with eigenvalue $k + n$. It is important that on $\widehat{\mathcal{H}}_{\mathbb{C}^{+i},\,k}$ the parameter k can take any value > 0, contrary to $\mathcal{H}_{\mathbb{C}^{+i},\,k}$ where k can take only the values $1/2, 1, 3/2, 2, \ldots$.

5.6 Unitary representations on the Hilbert space $L^2(\mathbb{R}^+, dt)$

The measure $dt/(2t)^{2k-1}$ in the scalar product (97) and the form of the eigenfunctions (98) strongly suggest to introduce the unitary mapping

$$V_k: \qquad \hat{f}(t) \to \check{f}(t) = \hat{f}(t)(2t)^{1/2-k} \quad (101)$$

of $L^2(\mathbb{R}^+, (2t)^{1-2k}\,dt)$ onto $\widehat{\mathcal{H}}_{+,\,k}$, the standard Hilbert space $L^2(\mathbb{R}^+, dt)$ on the positive real line with the standard orthonormal basis

$$f_{k,n}(t) = \sqrt{\frac{2\Gamma(n+1)}{\Gamma(2k+n)}}(2t)^{k-1/2}e^{-t}L_n^{2k-1}(2t) = i^{2k-n}\check{e}_{k,n}(t)\ . \tag{102}$$

The operator $\check{K}_3$ here has the form

$$\check{K}_3 = -\frac{1}{2}(t\frac{d^2}{dt^2} + \frac{d}{dt}) + \frac{1}{2}t + \frac{(2k-1)^2}{8t}\ , \tag{103}$$

with the property

$$\check{K}_3 f_{k,n}(t) = (k+n)f_{k,n}(t)\ . \tag{104}$$

5.7 Relationship to the radial wave functions of the hydrogen atom

If we define $f_{k,n} = t^{1/2}h_{k,n}(t)$, then the eigenvalue equation (104) can be rewritten as

$$-\frac{1}{2}\left(h''_{k,n} + \frac{2}{t}h'_{k,n}(t)\right) + \left(\frac{k(k-1)}{2t^2} - \frac{k+n}{t}\right)h_{k,n}(t) = -\frac{1}{2}h_{k,n}(t)\ . \tag{105}$$

This is just the radial Schrödinger Eq. for the hydrogen atom in 3 space dimensions with mass $m = 1$, angular momentum $k = l + 1$, fine structure constant $\alpha = k + n$ and bound state energy $E_{l,\,n_r} = -1/2$. As

$$E_{l,\,n_r} = -\frac{1}{2}\frac{\alpha^2}{(l+n_r+1)^2} \tag{106}$$

for the energy levels of the hydrogen atom, we see that our quantum number n here is to be identified with the radial quantum number $n_r = 0, 1, \ldots$.

Thus, for $k = 1, 2, \ldots$, we have related the quantum theory of the Schwarzschild black hole to that of the hydrogen atom with varying fine structure constant. The irreducible representation with $k = 1$ corresponds to the s-wave bound states of the hydrogen atom.

As in the gravitational case $\alpha \propto M^2$, we see that we are consistent here with the relation (2).

5.8 Unitary representations on the Hardy spaces of the upper half plane and the unit circle

Next we give the form of the eigenfunctions of K_3 in the Hardy spaces H^2_{+i} and H^2_+ with the scalar products (94) and (67) respectively:

We have to apply the inverse Fourier transformation (96) to the functions $\check{f}_{k,n}(t)$ with real $w = u$ and in this context use the relations [32]

$$\begin{aligned}
&\int_0^\infty e^{-pt} t^{k-1/2} L_n^{2k-1}(t)dt = \\
= \;& \frac{\Gamma(2k+n)\Gamma(k+1/2)}{\Gamma(n+1)\Gamma(2k)} p^{-k-1/2} F\left(-n, k+\frac{1}{2}; 2k; \frac{1}{p}\right) && (107) \\
= \;& \frac{\Gamma(k+n+1/2)}{\Gamma(n+1)} (p-1)^n p^{-n-k-1/2} \times && (108) \\
& \times F\left(-n, k-\frac{1}{2}; \frac{1}{2}-k-n; \frac{p}{p-1}\right) \; , \\
& p = \frac{1}{2}(1-iu) \; ,
\end{aligned}$$

where

$$\begin{aligned}
F(a,b;c;z) \;=\;& 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots && (109) \\
& + \frac{a(a+1)\cdots(a+\nu-1)b(b+1)\cdots(b+\nu-1)}{c(c+1)\cdots(c+\nu-1)} \frac{z^\nu}{\nu!} \cdots
\end{aligned}$$

is the hypergeometric series [33] which here is a polynomial of degree n because $a = -n$. Since [34]

$$\Gamma(k+\frac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2k)}{2^{2k-1}\Gamma(k)} \; , \tag{110}$$

we get the orthonormal system of eigenfunctions $\tilde{f}_{k,n}(u)$ of K_3 on H^2_{+i}:

$$\begin{aligned}
\tilde{f}_{k,n}(u) \;=\;& 2^{-k+1/2} \sqrt{\frac{\Gamma(2k+n)}{\Gamma(n+1)}} \frac{\Gamma(2k)}{\Gamma(k)} (1-iu)^{-k-1/2} \times \\
& \times F\left(-n, k+\frac{1}{2}; 2k; \frac{2}{1-iu}\right) \; , && (111) \\
& (1-iu)^k = (1+u^2)^{k/2} e^{ik\phi}, -\frac{\pi}{2} < \phi = -\arctan(u) < +\frac{\pi}{2} \; .
\end{aligned}$$

This set of orthonormal functions on $H^2_{+i} = L^2(\mathbb{R}, du)$ can be interpreted in the framework of orthogonal polynomials [35] in the following manner: For any $k > 0$ define the weight function

$$\tilde{w}_k(u) = \frac{\Gamma^2(2k)}{2^{2k-1}\Gamma^2(k)} (1+u^2)^{k-1/2} \tag{112}$$

and the polynomials of degree n:

$$\tilde{b}_{k,n}(u) = \sqrt{\frac{\Gamma(2k+n}{\Gamma(n+1)}}\; F\left(-n, k+\frac{1}{2}; 2k; \frac{2}{1-iu}\right) \; . \tag{113}$$

Then the scalar product (94) may written as

$$(\tilde{f}_{k,n_1}, \tilde{f}_{k,n_2})_{+i} = \int_{-\infty}^{\infty} du\; \tilde{w}_k(u)\; \bar{\tilde{b}}_{k,n_1}(u)\; \tilde{b}_{k,n_2}(u) = \delta_{n_1 n_2} \; . \tag{114}$$

The operator K_3 now is no longer a pure differential operator, but due to the term $\propto t^{-1}$ in Eq. (103), an integro-differential operator on H^2_{+i}.
In order to get the eigenfunctions $f_{k,n}(\phi)$ of K_3 in H^2_+ on the unit circle we have to follow up the inverse Fourier transformation (108) from above by the mapping — see Eq. (82) —

$$E^{-1}_{1/2} : f(z) = 2\sqrt{\pi}\; (z-i)^{-1} \tilde{f}\left(w = u = \frac{1-iz}{z-i}\right) \;, \; |z| = 1 \; . \tag{115}$$

Observing that $p/(p-1) = -1/(iz)$ and using the relations [33]

$$z^{-a} F(a, a-c+1; a-b+1; \frac{1}{z}) = F(a,b;c;z) \tag{116}$$

and (108), we then finally get for $k \geq 1/2$

$$\begin{aligned} f_{k,n}(\phi) &= \gamma_{k,n}\,(1-e^{i\phi})^{k-1/2}\; F\left(-n, k+\frac{1}{2}; \frac{3}{2}-k-n; e^{i\phi}\right) \;, \\ \gamma_{k,n} &= i\frac{\Gamma(n+k+1/2)}{\sqrt{\Gamma(n+1)\Gamma(2k+n)}} \;, \end{aligned} \tag{117}$$

where we have put $-iz = \exp(i\phi)$ because $|z| = 1$.
We now may proceed as before: As

$$(1-e^{i\phi})(1-e^{-i\phi}) = 2(1-\cos\phi) = 4\sin^2(\phi/2)$$

we may define the weight

$$w_k(\phi) = 2^{2k-1} \sin^{2k-1}(\phi/2) \tag{118}$$

and the orthogonal polynomials

$$b_{k,n}(\phi) = \gamma_{k,n}\; F\left(-n, k+\frac{1}{2}; \frac{3}{2}-k-n; e^{i\phi}\right) \tag{119}$$

for all $k > 0$ and can then write the scalar product (67) as

$$(f_{k,n_1}, f_{k,n_2})_+ = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, w_k(\phi)\, \overline{b}_{k,n_1}(\phi)\, b_{k,n_2}(\phi) = \delta_{n_1 n_2}\ . \tag{120}$$

We repeat the basic difference between the eigenfunctions (70) and (117) of the self-adjoint generator K_3 of the corresponding unitary representations of the compact subgroup of $SO^{\uparrow}(1,2)$ both sets of which belong to the same vector space: The set (70) belongs to the representations which are unitary with respect to the scalar product (69) whereas the set (117) is associated with the scalar product (67). Both representations are unitarily equivalent: this follows from the sequence of mappings we have been using and which are all unitary.

5.9 Other unitary representations on the Hardy space H_+^2 of the unit circle

One can implement the constituting relations (A.60)–(A.62) for a unitary representation on the Hardy space H_+^2 with the scalar product (67) and the basis $\exp(i\,n\,\phi)$, $n \in \mathbb{N}_0$, by chosing for the "ladder" operators $K_\pm$ "nonlocal" expressions [20]:

$$\breve{K}_3 = k + \frac{1}{i}\frac{d}{d\phi}\ , \tag{121}$$

$$\breve{K}_+ = e^{i\,\phi}\left[\left(2k + \frac{1}{i}\frac{d}{d\phi}\right)\left(1 + \frac{1}{i}\frac{d}{d\phi}\right)\right]^{1/2}\ ,\ \breve{K}_- = (\breve{K}_+)^+\ , \tag{122}$$

because

$$\breve{K}_+ e^{i\,n\,\phi} = [(2k+n)(n+1)]^{1/2}\, e^{i\,(n+1)\,\phi}\ . \tag{123}$$

5.10 Unitary representations in the state space of two harmonic oscillators

Finally we mention that all irreducible unitary representations of the positive discrete series of $SU(1,1)$ with $k = 1/2, 1, 3/2, 2, \ldots$ are contained in the tensor product of the Hilbert spaces of two harmonic oscillators [36, 37, 38], generated by creation and annihilation operators

$$[a_i, a_j^+] = \delta_{ij}\ ,\quad [a_i, a_j] = 0\ ,\quad [a_i^+, a_j^+] = 0\ ,\quad i,j = 1,2\ . \tag{124}$$

The operators

$$K_3 = \frac{1}{2}(a_1^+ a_1 + a_2^+ a_2 + 1)\ ,\quad K_+ = a_1^+ a_2^+\ ,\quad K_- = a_1 a_2 \tag{125}$$

obey the commutation relations

$$[K_3, K_+] = K_+ \ , \quad [K_3, K_-] = -K_- \ , \quad [K_+, K_-] = -2K_3 \tag{126}$$

of the Lie algebra of $SO^{\uparrow}(1,2)$ and its covering groups. The "ground state" $|k;0\rangle$ is defined by the property

$$a_j|k;0\rangle = 0 \ , \quad j = 1, 2, \tag{127}$$

and the other normalized states by

$$|k; n_1, n_2\rangle = \frac{1}{\sqrt{n_1!\, n_2!}} (a_2^+)^{n_2}\, (a_1^+)^{n_1} |k;0\rangle \ , \quad n_j \in \mathbb{N}_0 \ . \tag{128}$$

Notice that K_3 is just half the sum of the two Hamilton operators $H_j = (a_j^+ a_j + 1/2), j = 1, 2,$ of the two harmonic oscillators.

The relation between the number pair (n_1, n_2) and the pair $(k, n), n \in \mathbb{N}_0$, characterizing a state in an irreducible representation is obtained as follows: First we have

$$K_3|k; n_1, n_2\rangle = \frac{1}{2}(n_1 + n_2 + 1)|k; n_1, n_2\rangle = (k + n)|k; n_1, n_2\rangle \tag{129}$$

and second we have for the Casimir operator $Q = K_1^2 + K_2^2 - K_3^2 = K_+K_- + K_3(1 - K_3)$:

$$\begin{aligned} Q|k; n_1, n_2\rangle &= \{n_1\, n_2 + \frac{1}{4}(1 + n_1 + n_2)(1 - n_1 - n_2)\}|k; n_1, n_2\rangle \\ &= k(1 - k)|k; n_1, n_2\rangle \ . \end{aligned} \tag{130}$$

Thus we have the two relations

$$k = \frac{1}{2} + \frac{1}{2}|n_1 - n_2| \ ; \quad n = \frac{1}{2}(n_1 + n_2) - \frac{1}{2}|n_1 - n_2| = \min\{n_1,\, n_2\} \ . \tag{131}$$

They show that in this construction only representations with half integer or integer positive k are realizable and that the relations (131) are symmetric in n_1 and n_2. The latter property means that, except for $k = 1/2$ where $n_1 = n_2$, each irreducible representation with fixed k occurs twice in the tensor product $\mathcal{H}_1^{osc} \otimes \mathcal{H}_2^{osc}$ (because $n_1 - n_2 = \pm(2k - 1)$) of the harmonic oscillator Hilbert spaces $\mathcal{H}_j^{osc}, j = 1, 2$, realized, e.g., by the orthogonal Hermite functions on $L^2(\mathbb{R}, dx)$. For $k = 1$ we have the two possibilities $e_{n_1}(x_1) \otimes e_{n_1+1}(x_2)$ or $e_{n_1}(x_1) \otimes e_{n_1-1}(x_2)$, where $e_{n_1}(x_1) \in \mathcal{H}_1^{osc}$ and $e_{n_2}(x_2) \in \mathcal{H}_2^{osc}$.

5.11 Reintroducing Planck's constant

Up to now we have set $\hbar = 1$. We restore it explicitly in the same way as in the case of the rotation group: We just multiply each operator K_j by $\hbar$. This corresponds to the fact that the operator $\hat{p} = K_3$ is canonically conjugate to the dimensionless angle variable φ. According to Eq. (12) the quantization of the horizon area is then given by

$$A_{D-2}(k;n) = (k+n)\, a_{D-2}\ , \quad a_{D-2} = \frac{32\pi^2}{\gamma\,(D-3)}\, l_{P,D}^{D-2}. \tag{132}$$

6 Conclusions

Our above results show that the original ansatz of Ref. [7] to associate the Bekenstein spectrum (1) or (2) with a *finite* time interval $\Delta \propto R_S(M)$ which precedes the collaps of the Schwarzschild system to a black hole can be put on more solid grounds: Implementing the finite time interval by M–dependent periodic boundary conditions leads to a phase space with symplectic form $d\varphi \wedge dp$ which is globally diffeomorphic to $S^1 \times \mathbb{R}^+$. Such a phase space can be quantized group theoretically by means of the group $SO^{\uparrow}(1,2)$ (or its covering groups).

The main advantage of this approach is that it provides a Hilbert space and the basic self-adjoint operators for quantized Schwarzschild black holes.

The crucial point is that the classical variable p is proportional to the area A_{D-2} of the black hole horizon in any space-time dimension $D \geq 4$ and that the self-adjoint operator $\hat{p}$ has a discrete spectrum in any irreducible unitary representation of $SO^{\uparrow}(1,2)$. As we want the spectrum to be positive — because the area A is a positive quantity — only the positive discrete series among the irreducible unitary representations has the required properties. It provides the spectrum

$$A_{D-2}(k;n) \propto k+n,\ n \in \mathbb{N}_0\ . \tag{133}$$

Here the number $k > 0$ mathematically characterizes the representation and physically the "remnant" area of the ground state. As the energy of the n-th level is given by — see Eq. (10) —

$$E_{k;n} = \alpha_D\,(k+n)^{(D-3)/(D-2)}\ E_{P,D}\ ,\ n \in \mathbb{N}_0\ , \tag{134}$$

the number k determines the ground state energy like the number $1/2$ in the case of the harmonic oscillator. The value of k depends on the representation

to be employed: For the "true" representations of $SO^{\uparrow}(1,2)$ themselves k can take only the values $1, 2, \ldots$ (corresponding to the s-, p-, etc. states of the hydrogen atom!). For the two-fold covering groups $SU(1,1)$ or $SL(2,\mathbb{R})$ k may assume the values $k = 1/2, 1, 3/2, 2, \ldots$ and for the universal covering group $\widetilde{SO^{\uparrow}(1,2)}$ k can be any real positive number. Physical arguments will have to select the right value. If $k = 1$ the ground state area quanta have the same value as those of the excited levels which would leave us with only one kind of area quanta, but this must not be so, as can be seen from the harmonic oscillator where the energy of the ground state is just half of the basic energy quantum $\hbar\omega$.

In Ref. [20] arguments will be presented which suggest that the possible values of k should be restricted to the interval $(0, 1]$.

As to the physics we see the following picture emerging: The *area* of the quantized Schwarzschild black hole is built up *additively* and equidistantly from basic quanta whereas the *energy* (134) behaves differently: the energy of the n-th level may be interpreted [15, 13, 6] as the surface energy of a "bubble" of n area quanta.

It is an interesting and supporting result that the eigenvalues of the area operator in the spherically symmetric sector of loop quantum gravity (without matter) in 3+1 space-time dimensions for large n are proportional to n, too [39].

As to the degeneracies of the states one sees immediately from the formulae above that the eigenstates $e_{k,n}$ etc. are not degenerate in an irreducible representation.

One might think about passing to reducible representations, e.g. in terms of Fock spaces constructed from "1-particle" wave functions discussed above (2nd quantization). The operator K_3 then becomes a sum of the corresponding "irreducible" K_3. The degeneracies of the associated eigenvalues are then given by the possible partitions of a positive number n into smaller ones. For large n this yields [40] $d_n \sim g^{\sqrt{n}}$, $g > 1$, in contrast to g^n required to yield the correct Bekenstein-Hawking entropy (see Eq. (133)).

Yet one gets the desired thermodynamics, if, as one of us has proposed [6], each area quantum is assigned two degrees of freedom corresponding to the two possible orientations of a (classical) sphere. The energy spectrum (134) together with the degeneracy 2^n of the n-th level then lead [13, 6] to the Hawking temperature and the Bekenstein-Hawking entropy of a Schwarzschild black hole!

Thus, altogether a quite coherent picture of the quantum theory of Schwarzschild black holes and their thermodynamics emerges.

The groups $SO^{\uparrow}(1,2)$, $SL(2,\mathbb{R})$ etc. and their Lie algebra have been playing a number of roles in the context of recent attempts to quantize black holes:

Hollmann [41] has analyzed the quantum theory of Schwarzschild (Taub-NUT) black holes in terms of the coset space $SL(2,\mathbb{R})/SO(2)$ which yields a continuous spectrum, whereas we use the coset space $SL(2,\mathbb{R})/\mathrm{N}$, where N is the nilpotent group from an Iwasawa decomposition.

The group $SL(2,\mathbb{R})$ and its Lie algebra have a prominant role also in recent discussions of black holes in ($D = 2$)- and ($D = 3$)-dimensional models of quantum gravity, especially Anti-de Sitter spaces AdS_2 and AdS_3 (see Refs. [42, 43, 44] and the literature quoted there). In the 3-dimensional case the Lie algebra $\mathcal{L}SL(2,\mathbb{R})$ plays an essential role as the basic subalgebra of the associated Virasoro algebras [45, 46]. At the moment it is an open question whether and how these approaches are related to ours above.

See also the interesting application of the group $SL(2,\mathbb{R})$ to black holes by Gibbons and Townsend [47].

Acknowledgement

We thank N. Düchting for discussions.

Note added: After our paper was submitted as an e-print we became aware of an earlier group theoretical quantization of the symplectic manifold $S^1 \times \mathbb{R}^+$ in terms of the group $SO^{\uparrow}(1,2)$ by R. Loll [48] in a different context.

Appendices

A Properties of the group $SO^{\uparrow}(1,2)$, of some of its covering groups and their irreducible unitary representations of the positive discrete series

The purpose of the present appendix is to summarize the main properties of the group $SO^{\uparrow}(1,2)$ which are important for our discussion above, where this group, its covering groups, its Lie algebra and its irreducible unitary representations, especially those of the positive discrete series, have been employed as the quantizing framework for Schwarzschild black holes. Practically all of this appendix is contained in a wealth of literature about this group which is the most elementary of noncompact semisimple Lie groups. Potential readers of this paper, however, will find it convenient to have the required properties assembled in one unit.
The essential classical paper on the group $SO^{\uparrow}(1,2)$ and its irreducible unitary repesentations is (still!) that of Bargmann [49]. In the meantime there are a number of monographs which deal with the group $SO^{\uparrow}(1,2)$, its covering groups and their representations [50, 51, 52, 53, 54, 55, 56, 57, 38]. As these textbooks contain many references to the original literature we mention only the most essential ones for our purposes.

A.1 The group and some of its covering groups

In order to see the homomorphism between $SO^{\uparrow}(1,2)$ and its mutually isomorphic twofold covering groups $SU(1,1), SL(2,\mathbb{R})$ and the symplectic group $Sp(1,\mathbb{R})$ in 2 dimensions it is convenient to start from the action of the group $SL(2,\mathbb{C})$ — the twofold covering group of the proper orthochronous Lorentz group $SL^{\uparrow}(1,3)$ — on Minkowski space M^4 with the scalar product $x \cdot x = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$: Define the hermitean matrix

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \;,\;\; \mathrm{det}X = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \;. \quad \text{(A.1)}$$

If $C \in SL(2,\mathbb{C}),\ \mathrm{det}C = 1$, then

$$X \to \hat{X} = C \cdot X \cdot C^{+} \;,\;\; \mathrm{det}\hat{X} = \mathrm{det}X \;, \quad \text{(A.2)}$$

induces a proper orthochronous Lorentz transformation on M^4. Here C^+ means the hermitean conjugate of the matrix C.
Subgroups $SO^{\uparrow}(1,2)$ may be obtained by looking for those transformations

(A.2) which leave one of the coordinates $x^j,\ j = 1, 2$ or 3 fixed:
The transformations with the property

$$C \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot C^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.3}$$

leave the coordinates x^3 invariant and represent the subgroup $SU(1,1) = \{g_0\} \subset SL(2,\mathbb{C})$:

$$g_0 = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \ , \quad |\alpha|^2 - |\beta|^2 = 1 \ . \tag{A.4}$$

$\bar{\alpha}$: complex conjugate of α. If we let g_0 act on a 2-dimensional complex vector space, then

$$g_0 \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} \ , \quad |\hat{z}_1|^2 - |\hat{z}_2|^2 = |z_1|^2 - |z_2|^2 \ . \tag{A.5}$$

If now $|z_2| > |z_1|$ and $z = z_1/z_2$ then $SU(1,1)$ maps the interior $\mathcal{D} = \{z; |z| < 1\}$ of the unit disc in the complex z-plane (transitively) onto itself:

$$z \to \hat{z} = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \ . \tag{A.6}$$

If we write $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$, then $|\alpha|^2 - |\beta|^2 = \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 = 1$. This means that the group manifold of $SU(1,1)$ is homeomorphic to the 3-dimensional Anti-de Sitter space [58] AdS_3.
The subgroup of $SL(2,\mathbb{C})$ with the property

$$C \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot C^+ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{A.7}$$

leaves the coordinates x^2 invariant. It constitutes the group $SL(2,\mathbb{R})$:

$$C = g_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \ , \ a_{jk} \in \mathbb{R} \ , \ \det g_1 = 1 \ . \tag{A.8}$$

As

$$g_1 \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot g_1^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{A.9}$$

the group $SL(2,\mathbb{R})$ is identical with the real symplectic group $Sp(1,\mathbb{R})$ in 2 dimensions.
The unitary matrix

$$C_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \ , \ \det C_0 = 1 \ , \ C_0^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = C_0^+ \ , \tag{A.10}$$

has the property

$$C_0 \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} C_0^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{(A.11)}$$

and therefore implements an isomorphism between $SU(1,1)$ and $SL(2,\mathbb{R})$:

$$C_0 \cdot g_0 \cdot C_0^{-1} = g_1 \ . \qquad \text{(A.12)}$$

It is obvious that the isomorphic groups $SU(1,1), SL(2,\mathbb{R})$ and $Sp(1,\mathbb{R})$ are twofold covering groups of $SO^{\uparrow}(1,2)$.
The group $SL(2,\mathbb{R})$ maps the complex upper half plane $\mathbb{C}^{+i} = \{z = x + iy,\ y > 0\}$ transitively onto itself:

$$z \to \hat{z} = \frac{a_{11}z + a_{12}}{a_{21}z + a_{22}} \ , \quad \Im(\hat{z}) = \frac{y}{(a_{22} + a_{21}x)^2 + a_{21}^2 y^2} \ . \qquad \text{(A.13)}$$

Of special interest for our purposes is the (unique!) Iwasawa decomposition [59, 56] of the groups $\mathrm{G}_1 \equiv SL(2,\mathbb{R})$ and $\mathrm{G}_0 \equiv SU(1,1)$: $\mathrm{G}_1 \equiv \mathrm{K}_1 \cdot \mathrm{A}_1 \cdot \mathrm{N}_1$, $\mathrm{G}_0 = \mathrm{K}_0 \cdot \mathrm{A}_0 \cdot \mathrm{N}_0$, where K is the maximal compact subgroup, A a maximally abelian noncompact subgroup and N a nilpotent group. For G_1 this decomposition is

$$\mathrm{K}_1 : \quad k_1 = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \ , \ \theta \in (-2\pi, +2\pi] \ , \qquad \text{(A.14)}$$

$$\mathrm{A}_1 : \quad a_1 = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \ , \ t \in \mathbb{R} \ , \qquad \text{(A.15)}$$

$$\mathrm{N}_1 : \quad n_1 = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \ , \ \xi \in \mathbb{R} \ . \qquad \text{(A.16)}$$

Each element g_1 has a unique decomposition $g_1 = k_1 \cdot a_1 \cdot n_1$. The isomorphism (12) gives the corresponding decomposition of G_0:

$$\mathrm{K}_0 : \quad k_0 = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \ , \ \theta \in (-2\pi, +2\pi] \ , \qquad \text{(A.17)}$$

$$\mathrm{A}_0 : \quad a_0 = \begin{pmatrix} \cosh(t/2) & -i\sinh(t/2) \\ i\sinh(t/2) & \cosh(t/2) \end{pmatrix} \ , \ t \in \mathbb{R} \ , \qquad \text{(A.18)}$$

$$\mathrm{N}_0 : \quad n_0 = \begin{pmatrix} 1 - i\xi/2 & \xi/2 \\ \xi/2 & 1 + i\xi/2 \end{pmatrix} \ , \ \xi \in \mathbb{R} \ . \qquad \text{(A.19)}$$

In addition to the above subgroups the following two ones are of interest to us:

$$\mathrm{B}_1: \qquad b_1 = \begin{pmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{pmatrix}, \ s \in \mathbb{R}, \tag{A.20}$$

$$\mathrm{B}_0: \qquad b_0 = C_0^{-1} \cdot b_1 \cdot C_0 = \begin{pmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{pmatrix}, \tag{A.21}$$

$$\bar{\mathrm{N}}_1: \qquad \bar{n}_1 = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix}, \ \xi \in \mathbb{R}, \tag{A.22}$$

$$\bar{\mathrm{N}}_0: \qquad \bar{n}_0 = \begin{pmatrix} 1 + i\xi/2 & \xi/2 \\ \xi/2 & 1 - i\xi/2 \end{pmatrix}. \tag{A.23}$$

Two more decompositions of $SL(2,\mathbb{R})$ or $SU(1,1)$ are important for the construction of their unitary representations:
Cartan (or "polar") decomposition [59, 56]:
Each element of $SL(2,\mathbb{R})$ can be written as

$$g_1 = k(\theta_2) \cdot a_1(t) \cdot k(\theta_1), \tag{A.24}$$

where $a_1(t)$ is determined uniquely and $k(\theta_1), k(\theta_2)$ up to a relative sign, that is up to the center $\mathbb{Z}_2$ of $SL(2,\mathbb{R})$.
Bruhat decomposition [59, 52, 53]:
From

$$k(\theta) \cdot a_1(t) \cdot k(-\theta) = \tag{A.25}$$
$$\begin{pmatrix} \cos^2(\theta/2)e^{t/2} + \sin^2(\theta/2)e^{-t/2} & \sin(\theta/2)\cos(\theta/2)(e^{-t/2} - e^{t/2}) \\ \sin(\theta/2)\cos(\theta/2)(e^{-t/2} - e^{t/2}) & \cos^2(\theta/2)e^{-t/2} + \sin^2(\theta/2)e^{t/2} \end{pmatrix}$$

one sees that

$$k(\theta) \cdot a_1(t) \cdot k(-\theta) \ = \ a_1(t) \text{ for } \theta = 0, 2\pi, \tag{A.26}$$
$$k(\theta) \cdot a_1(t) \cdot k(-\theta) \ \subset \ \mathrm{A}_1 \text{ for } \theta = 0, \pm\pi, 2\pi, \tag{A.27}$$

which means that the centralizer $C_{\mathrm{K}_1}(\mathrm{A}_1)$ and normalizer $N_{\mathrm{K}_1}(\mathrm{A}_1)$ of A_1 in K_1 are given by

$$C_{\mathrm{K}_1}(\mathrm{A}_1) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \mathbb{Z}_2, \tag{A.28}$$

$$N_{\mathrm{K}_1}(\mathrm{A}_1) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}. \tag{A.29}$$

The quotient group

$$\begin{aligned} W &= N_{K_1}(\mathrm{A}_1)/\mathbb{Z}_2 & \text{(A.30)}\\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathbb{Z}_2,\ w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bmod \mathbb{Z}_2 \right\} \end{aligned}$$

is called the Weyl-group of $SL(2,\mathbb{R})$. Its associated Bruhat decomposition of $SL(2,\mathbb{R})$ is

$$\mathrm{G}_1 = \mathbb{Z}_2 \cdot \mathrm{A}_1 \cdot \mathrm{N}_1 \cup \mathrm{N}_1 \cdot w \cdot \mathbb{Z}_2 \cdot \mathrm{A}_1 \cdot \mathrm{N}_1\ ,\ w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ . \tag{A.31}$$

Here $\mathbb{Z}_2 \cdot \mathrm{A}_1$ is the group

$$\mathrm{D}_1 = \mathbb{Z}_2 \cdot \mathrm{A}_1 = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}\ ,\ c \in \mathbb{R} - \{0\} \right\}\ . \tag{A.32}$$

The relation (A.31) means that each element of $SL(2,\mathbb{R})$ may be decomposed uniquely either as an element of the "parabolic" subgroup $\mathrm{P}_1 = \mathrm{D}_1 \cdot \mathrm{N}_1$ or as an element of $\mathrm{N}_1 \cdot w \cdot \mathrm{P}_1$.
The Bruhat decomposition of $SL(2,\mathbb{R})$ plays a central role in Sally's construction [51] of the irreducible unitary representations of the universal covering group $\widetilde{SL(2,\mathbb{R})}$.
As the compact subgroup K_1, or K_0, $\simeq S^1$, is infinitely connected, the groups $SL(2,\mathbb{R})$ and $SU(1,1)$ have an infinitely sheeted universal covering group which, according to Bargmann, may be parametrized as follows: Starting from $SU(1,1)$ define

$$\begin{aligned} \gamma &= \beta/\alpha\ ,\ |\alpha|^2 - |\beta|^2 = 1\ (\Rightarrow |\gamma| < 1);\quad \omega = \arg(\alpha)\ ; & \text{(A.33)}\\ \alpha &= e^{i\omega}(1 - |\gamma|^2)^{-1/2}\ ,\ |\gamma| < 1\ ,\ \beta = e^{i\omega}\gamma(1 - |\gamma|^2)^{-1/2}\ . & \text{(A.34)} \end{aligned}$$

Then

$$\begin{aligned} SU(1,1) &= \{g_0 = (\omega, \gamma),\ \omega \in (-\pi, \pi],\ |\gamma| < 1\}\ , & \text{(A.35)}\\ \tilde{\mathrm{G}} \equiv \widetilde{SU(1,1)} &= \widetilde{SL(2,\mathbb{R})} = \{\tilde{g} = (\omega, \gamma),\ \omega \in \mathbb{R},\ |\gamma| < 1\}\ . & \text{(A.36)} \end{aligned}$$

The group composition law for $\tilde{g}_3 = \tilde{g}_2 \cdot \tilde{g}_1$ is given by

$$\begin{aligned} \gamma_3 &= (\gamma_1 + \gamma_2 e^{-2i\omega_1})(1 + \bar{\gamma}_1\gamma_2 e^{-2i\omega_1})^{-1}\ , & \text{(A.37)}\\ \omega_3 &= \omega_1 + \omega_2 + \frac{1}{2i}\ln[(1 + \bar{\gamma}_1\gamma_2 e^{-2i\omega_1})(1 + \gamma_1\bar{\gamma}_2 e^{2i\omega_1})^{-1}]\ . & \text{(A.38)} \end{aligned}$$

A.2 Lie algebra

As the structure of the 3-dimensional Lie algebra $\mathcal{L}SO^{\uparrow}(1,2) = \{l\}$ of $SO^{\uparrow}(1,2)$ is the same as that of all its covering groups we may calculate it by using any of them. For $SL(2,\mathbb{R})$ we get from the subgroups (A.14)-(A.16), (A.20) and (A.22):

$$l_{K_1} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ l_{A_1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ l_{B_1} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \tag{A.39}$$

$$l_{N_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ ,\ \ l_{\bar{N}_1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ , \tag{A.40}$$

which are not independent (in the following we skip the indices "1" or "2", because the structure relations are independent of them):

$$l_N + l_{\bar{N}} = 2\, l_B\ ,\quad l_N - l_{\bar{N}} = 2\, l_K\ \ . \tag{A.41}$$

We have the commutation relations

$$[l_K, l_A] \;=\; -l_B\ ,\ \ [l_K, l_B] = l_A\ ,\ \ [l_A, l_B] = l_K\ , \tag{A.42}$$

$$[l_K, l_N] = l_A\ ,\ \ [l_K, l_{\bar{N}}] = l_A\ , \tag{A.43}$$

$$[l_A, l_N] = l_N\ ,\ \ [l_A, l_{\bar{N}}] = -l_{\bar{N}}\ , \tag{A.44}$$

$$[l_B, l_N] = -l_A\ ,\ \ [l_B, l_{\bar{N}}] = l_A\ , \tag{A.45}$$

$$[l_N, l_{\bar{N}}] = 2\, l_A\ . \tag{A.46}$$

The relations (A.42) show that the algebra is semisimple, the first of the Eqs. (A.44) that A and N combined form a 2-dimensional subgroup and that A is a normalizer of N.

A.3 Irreducible unitary representations of the positive discrete series

As the group $SO^{\uparrow}(1,2)$ is noncompact, its irreducible unitary representations are infinite-dimensional. Their structure can be seen already from its Lie algebra: In unitary representations the elements $-il_K, -il_A, -il_B$ of the Lie algebra correspond to self-adjoint operators K_3, K_1, K_2 which obey die commutation relations

$$[K_3, K_1] = iK_2\ ,\ \ [K_3, K_2] = -iK_1\ ,\ \ [K_1, K_2] = -iK_3\ , \tag{A.47}$$

or, with the definitions

$$K_+ = K_1 + iK_2\ ,\ \ K_- = K_1 - iK_2\ , \tag{A.48}$$

$$[K_3, K_+] = K_+ \ , \quad [K_3, K_-] = -K_- \ , \quad [K_+, K_-] = -2K_3 \ . \tag{A.49}$$

The relations (A.47) are invariant under the replacement $K_1 \to -K_1, K_2 \to -K_2$ and the relations (A.49) invariant under $K_+ \to \omega K_+, K_- \to \bar{\omega} K_-$, where $|\omega| = 1$. These relations are in addition invariant under the transformations $K_+ \leftrightarrow K_-, K_3 \to -K_3$.

In irreducible unitary representations the operator K_- is the adjoint operator of K_+ : $(f_1, K_+ f_2) = (K_- f_1, f_2)$, and vice versa, where it is assumed that f_1, f_2 belong to the domains of definition of K_+ and K_-.

The Casimir operator Q of a representation is defined by

$$Q = K_1^2 + K_2^2 - K_3^2 \tag{A.50}$$

and we have the relations

$$K_+ K_- = Q + K_3(K_3 - 1) \ , \quad K_- K_+ = Q + K_3(K_3 + 1) \ . \tag{A.51}$$

All unitary representations make use of the fact that K_3 is the generator of a compact group and that its eigenfunctions g_m are normalizable elements of the associated Hilbert space $\mathcal{H}$.

The relations (A.49) imply

$$K_3 \, g_m = m \, g_m \ , \tag{A.52}$$
$$K_3 K_+ g_m = (m+1) K_+ g_m \ , \tag{A.53}$$
$$K_3 K_- g_m = (m-1) K_- g_m \ , \tag{A.54}$$

which, combined with Eqs. (A.51), lead to

$$(g_m, K_+ K_- g_m) = (K_- g_m, K_- g_m) = q + m(m-1) \geq 0 \ , \tag{A.55}$$
$$(g_m, K_- K_+ g_m) = q + m(m+1) \geq 0 \ , \quad q = (g_m, Q g_m), \tag{A.56}$$

implying

$$(K_+ g_m, K_+ g_m) = 2m + (K_- g_m, K_- g_m) \geq 0. \tag{A.57}$$

In the following we assume that we have an irreducible representation for which the functions g_m are eigenfunctions of the Casimir operator Q, too: $Q g_m = q \, g_m$.

The relations (A.52)-(A.54) show that the eigenvalues of K_3 in principle can be any real number, where, however, different eigenvalues differ by an integer. For the "principle" and the "complementary" series the spectrum of K_3 is unbounded from below and above .

The positive discrete series D_+ of irreducible unitary representations is characterized by the property that there exists a lowest eigenvalue $m = k$ such that

$$K_- g_k = 0 \ . \tag{A.58}$$

Then the relations (A.55)-(A.57) imply

$$q = k(1-k) \ , \quad k > 0 \ , \quad m = k + n, \ n = 0, 1, 2, \dots . \tag{A.59}$$

The relations (A.52)-(A.54) now take the form

$$\begin{aligned} K_3 g_{k,n} &= (k+n) g_{k,n} \ , & \text{(A.60)} \\ K_+ g_{k,n} &= \omega_n \left[(2k+n)(n+1)\right]^{1/2} g_{k,n+1} \ , \quad |\omega_n| = 1 \ , & \text{(A.61)} \\ K_- g_{k,n} &= \frac{1}{\omega_{n-1}} [(2k+n-1)n]^{1/2} g_{k,n-1} \ . & \text{(A.62)} \end{aligned}$$

The phases ω_n guarantee that $(f_1, K_+ f_2) = (K_- f_1, f_2)$.
Up to now we have allowed for any value of $k > 0$. It turns out [26, 51] that this is so for the irreducible representations of the universal covering group $\widetilde{SO^\uparrow}(1,2)$. These representations may be realized for $k \geq 1/2$ in the Hilbert space of holomorphic functions on the unit disc $\mathcal{D} = \{z, |z| < 1\}$ with the scalar product

$$(f,g)_k = \frac{2k-1}{\pi} \int_{\mathcal{D}} \bar{f}(z) g(z) (1 - |z|^2)^{2k-2} dx dy \ . \tag{A.63}$$

as

$$\begin{aligned} [U(\tilde{g},k)f](z) &= e^{2ik\omega} (1 - |\gamma|^2)^k (1 + \bar{\gamma} z)^{-2k} f\left(\frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}\right) \ , & \text{(A.64)} \\ \tilde{g} &= (\omega, \gamma) \ , \quad \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = h(\tilde{g}) \ \in SU(1,1) \ . & \text{(A.65)} \end{aligned}$$

Because $|\gamma\, z| < 1$, the function $(1 + \bar{\gamma} z)^{-2k}$ is, for $k > 0$, defined in terms of a series expansion.

For $SU(1,1)$ we have $\omega \in \mathbb{R} \bmod 2\pi$. Uniqueness of the phase factor then requires $k = 1/2, 1, 3/2, \cdots$.

For $SO^\uparrow(1,2)$ itself we have $\omega \in \mathbb{R} \bmod \pi$ which implies $k = 1, 2, \cdots$.

More about the unitary representations can be found in chapter 5 of the main text.

B Reduced phase space of D-dimensional Schwarzschild gravitational systems

B.1 Symplectic reduction

We start from the spherically symmetric ADM line-element

$$ds^2 = -N^2 dt^2 + P^2(dr + N_r dt)^2 + Q^2 d\Omega^2 \ , \tag{B.1}$$

where r is the radial coordinate and $d\Omega^2$ the line element of a unit sphere S^{D-2}, embedded in $(D-1)$-dimensional flat space. The functions P, Q, N and N_r depend only on r and t.

The line element of the two-dimensional radial manifold with coordinates $(x^0, x^1) = (t, r)$ will be denoted by

$$d\sigma^2 = g_{ij} dx^i dx^j = -N^2 dt^2 + P^2(dr + N_r dt)^2. \tag{B.2}$$

The line element ds^2 is conformally equivalent to

$$\begin{aligned} d\hat{s}^2 \ &= \ d\hat{\sigma}^2 + d\Omega^2 \ , \quad d\hat{\sigma}^2 = Q^{-2} d\sigma^2 \ , \\ & \quad\ d\Omega^2 = g_{AB}\, dy^A dy^B, \ 2 \le A, B \le D-1 \ , \end{aligned} \tag{B.3}$$

where $d\Omega^2$ is the line element of a $(D-2)$-dimensional space of constant positive curvature 1. Its Ricci curvature tensor is given by $R^{(D-2)}_{AB} = \Lambda g_{AB}$. A direct calculation using, e. g., the special form

$$d\Omega^2 = \frac{\delta_{CD}\, y^C y^D}{(1 + \frac{1}{4}\delta_{AB}\, y^A y^B)^2}$$

shows that $\Lambda = D - 3$.

In order to perform a symplectic reduction we have to insert the metric (B.1) into the Einstein-Hilbert-action. This has already been done, for a different purpose, in Ref. [60] with the result that the spherically symmetric theory is described by a two-dimensional dilaton action

$$S = \int dt dr \sqrt{|\det \tilde{g}_{ij}|}\left(\phi \tilde{R} - V(\phi)\right). \tag{B.4}$$

The metric $\tilde{g}_{ij}$ is obtained from g_{ij} by the conformal transformation

$$d\tilde{\sigma}^2 = \tilde{g}_{ij} dx^i dx^j = Q^{D-3} d\sigma^2 \ , \tag{B.5}$$

and the dilaton field ϕ is introduced by

$$\phi = Q^{D-2} \ , \tag{B.6}$$

with potential

$$V(\phi) = -(D-2)\Lambda\phi^{-(D-2)^{-1}} = -(D-2)(D-3)\phi^{-(D-2)^{-1}} \ . \qquad \text{(B.7)}$$

Arrived at a two-dimensional dilaton action we can adopt results of the symplectic reduction of Ref. [10]. There the line element is parameterized as

$$d\tilde{\sigma}^2 = e^{2\rho}(-\sigma^2 dt^2 + (dr + L\,dt)^2) \qquad \text{(B.8)}$$

and the canonical variables are $(\rho, P_\rho; \phi, P_\phi; L, P_L; \sigma, P_\sigma)$ subject to the constraints

$$\begin{aligned} P_L &\approx 0 \ , && \text{(B.9)}\\ P_\sigma &\approx 0 \ , && \text{(B.10)}\\ \mathcal{E} := \rho' P_\rho + \phi' P_\phi - P_\rho' &\approx 0 \ , && \text{(B.11)}\\ \mathcal{G} := 2\phi'' - 2\phi'\rho' - \frac{1}{2}P_\phi P_\rho + e^{2\rho}V(\phi) &\approx 0 \ . && \text{(B.12)} \end{aligned}$$

A prime denotes differentiation with respect to r, whereas a dot will denote differentiation with respect to t.

The first two constraints eliminate the variables L and σ, whereas the remaining two are equivalent to $\mathcal{E} \approx 0$ and $C' \approx 0$ with

$$C := e^{-2\rho}\left(\frac{1}{4}P_\rho^2 - (\phi')^2\right) - j(\phi) \ , \quad \frac{dj}{d\phi} = V(\phi) \ . \qquad \text{(B.13)}$$

C is a physical observable, which is forced by the constraints to be spatially constant. It represents the only physical degree of freedom.
The last step consists in connecting the results of the two quoted papers [10, 60] by the transformation of variables

$$\begin{aligned} \rho &\mapsto P = e^\rho \phi^{(3-D)/(2D-4)} \ , && \text{(B.14)}\\ \phi &\mapsto Q = \phi^{1/(D-2)} \ , && \text{(B.15)}\\ L &\mapsto N_r = L \ , && \text{(B.16)}\\ \sigma &\mapsto N = \sigma e^\rho \phi^{(3-D)/(2D-4)} \ . && \text{(B.17)} \end{aligned}$$

For the momenta we get

$$\begin{aligned} P_N &= 0 \ , && \text{(B.18)}\\ P_{N_r} &= 0 \ , && \text{(B.19)}\\ P_\phi &= \frac{1}{D-2}Q^{3-D}P_Q - \frac{1}{2}\frac{D-3}{D-2}PQ^{2-D}P_P \ , && \text{(B.20)}\\ P_\rho &= PP_P = 2(D-2)PQ^{D-3}N^{-1}(-\dot{Q} + N_r Q'). && \text{(B.21)} \end{aligned}$$

Using our $V(\phi)$ we have

$$j(\phi) = -(D-2)^2 \phi^{(D-3)/(D-2)} + k \ . \tag{B.22}$$

Here k is a constant of integration. Inserting all these formulae in Eq. (B.13) we obtain

$$C = \frac{1}{4} Q^{D-3} P_P^2 - (D-2)^2 Q^{D-3} \left((Q' P^{-1})^2 - 1 \right) - k. \tag{B.23}$$

In order to reveal its relation to the Schwarzschild mass M we compare it with the Schwarzschild line element in D space-time dimensions [14]: $Q = r$, $N_r = 0$, and

$$P^{-2} = N^2 = F := 1 - \frac{16\pi\, G\, M}{(D-2)\omega_{D-2}\, r^{D-3}} \ , \tag{B.24}$$

where G_D is the Newton constant, M the Schwarzschild (ADM) mass.

Due to $\dot{Q} = N_r = 0$ we have $P_P = 0$ (outside the horizon) and the resulting expression

$$C = -(D-2)^2\, r^{D-3}\, (F-1) - k = \frac{16\pi G_D\, M\, (D-2)}{\omega_{D-2}} - k \tag{B.25}$$

shows the relation of C to the Schwarzschild mass.

B.2 Reduced Hamiltonian

In order to determine the Hamiltonian associated with the observable C we have to add an appropriate surface term which renders the action functionally differentiable. For our functions $(\rho, P_\rho, \phi, P_\phi, L, \sigma)$ we choose fall-off conditions which are prescribed by asymptotic flatness of the Schwarzschild space-time. The remaining formulae remain, however, true for arbitrary dilaton potentials $V(\phi)$. For the functions (P, P_P, Q, P_Q, N, N_r) the fall-off conditions at $r \to \pm\infty$ (r is the radial coordinate of an asymptotically cartesian coordinate system) are

$$\begin{aligned}
Q &= |r| + O(|r|^{-\epsilon}), && \text{(B.26)}\\
P &= 1 + M_\pm(t)|r|^{3-D} + O(|r|^{3-D-\epsilon}), && \text{(B.27)}\\
P_Q &= O(|r|^{-1-\epsilon}), && \text{(B.28)}\\
P_P &= O(|r|^{-\epsilon}), && \text{(B.29)}\\
N_r &= O(|r|^{-\epsilon}), && \text{(B.30)}\\
N &= N_\pm(t) + O(|r|^{-\epsilon}). && \text{(B.31)}
\end{aligned}$$

From these conditions follow the fall-off conditions for the functions ρ, P_ρ, ϕ, P_ϕ, L, σ by using the transformation (B.14)-(B.17).

In order to extract the surface term we first perform, following [9], an appropriate canonical transformation which replaces the variable ρ by the observable C. The momentum conjugate to C is (locally)

$$P_C = -\frac{2e^{2\rho}P_\rho}{P_\rho^2 - 4(\phi')^2}. \tag{B.32}$$

The relation (B.32) becomes singular on the horizon. One can avoid this by defining a variable conjugate to C which is nowhere singular, e.g. in the framework of Poisson σ–models [61, 62], that is the singularity in Eq. (B.32) causes no problems here.

Because C and P_C do not depend on P_ϕ we have $\{C,\phi\} = 0 = \{P_C,\phi\}$ and we can use ϕ as a second variable. However, due to $\{P_\phi, C\} \neq 0$, we have to replace P_ϕ by a new momentum P_ψ conjugate to $\psi := \phi$. As in Ref. [9] it can be calculated by equating the generators of diffeomorphisms in both sets of variables:

$$\rho' P_\rho + \phi' P_\phi - P_\rho' = C' P_C + \phi' P_\psi \tag{B.33}$$

using the fact that both C and ϕ are scalars. The last equation yields

$$P_\psi = P_\phi + 4\frac{\phi' P_\rho' - \phi'' P_\rho}{P_\rho^2 - 4(\phi')^2} - 2\frac{e^{2\rho}P_\rho V(\phi)}{P_\rho^2 - 4(\phi')^2} = -2(P_\rho^2 - 4(\phi')^2)^{-1}(P_\rho \mathcal{G} + 2\phi' \mathcal{E}) \tag{B.34}$$

which shows that P_ψ is constrained to be zero.

Now we will show that the transformation $(\rho, P_\rho, \phi, P_\phi) \mapsto (C, P_C, \phi, P_\psi)$ is canonical by showing that the difference of the corresponding Liouville forms is exact. First, we have

$$\begin{aligned} &P_\rho\delta\rho + P_\phi\delta\phi - P_C\delta C - P_\psi\delta\phi \\ &= \delta\left(P_\rho + 2\phi' \log\left|\frac{2\phi' - P_\rho}{2\phi' + P_\rho}\right|\right) + 2\left(\delta\phi \log\left|\frac{2\phi' + P_\rho}{2\phi' - P_\rho}\right|\right)'. \end{aligned} \tag{B.35}$$

As a consequence of the fall-off conditions the integral over the last derivative term vanishes (at the horizon, i. e., at the singular points of the logarithm, the integral has to be interpreted as principal value). Therefore, the difference

$$\int dr(P_\rho\delta\rho + P_\phi\delta\phi - P_C\delta C - P_\psi\delta\phi) = \delta\int dr\left(P_\rho + 2\phi' \log\left|\frac{2\phi' - P_\rho}{2\phi' + P_\rho}\right|\right) \tag{B.36}$$

of the Liouville forms is exact, and the transformation to the variables (C, P_C, ϕ, P_ψ) is canonical. Assuming $\phi' \neq 0$, which will be fulfilled for Schwarzschild systems, the constraints $\mathcal{E} \approx 0$ and $\mathcal{G} \approx 0$ are equivalent to $C' \approx 0$ and $P_\psi \approx 0$. We use these constraints together with the new canonical variables and obtain the action

$$S = \int dtdr(P_C\dot{C} + P_\psi\dot{\phi} - N^C C' - N^\psi P_\psi) \ , \tag{B.37}$$

where N^C and N^ψ are new Lagrange multipliers. Because of the asymptotic relation $C' \sim \mp(D-2)\mathcal{G}$, which follows from the fall-off conditions, the asymptotic values of N^C and N are related by $N^C \sim \mp(D-2)^{-1}N$. The only spatial derivative in the new action appears in the term $-N^C C'$, and, therefore, the action can be made functionally differentiable by adding the surface term

$$\int dt(N^C_+ C_+ - N^C_- C_-) = -\int dt(D-2)^{-1}(N_+C_+ + N_-C_-). \tag{B.38}$$

$C_\pm(t)$, $N^C_\pm(t)$ and $N_\pm(t)$ are the values of C, N^C and N at $r \to \pm\infty$. This surface action and the formula (B.25) show that in any space-time dimension the reduced Hamiltonian of a spherically symmetric black hole is given by its mass (times the lapse function) after imposing the constraints: Rescaling S by $(16\pi G_D)^{-1}\omega_{D-2}$ to obtain the physical action in D dimensions, and setting $k = 0$ in Eq. (B.25), we finally obtain

$$H_{\rm red} = \frac{\omega_{D-2}}{16\pi\, G_D}(N_+ + N_-)\frac{C}{D-2} = (N_+ + N_-)\, M \tag{B.39}$$

with $C := C_+ = C_-$ (due to $C' = 0$).

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